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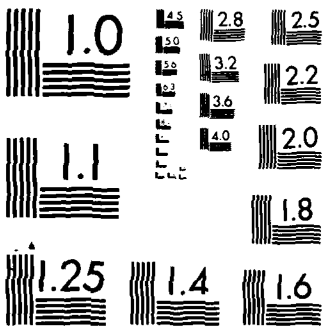

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ANALYSIS OF THE DYNAMIC BEHAVIOR OF AN INTENSE CHARGED PARTICLE BEAM USING THE SEMIGROUP APPROACH

DISSERTATION

Max A. Stafford
Maj USAF

AFIT/DS/ENC/85-1

DEPARTMENT OF THE AIR FORCE
AIR UNIVERSITY
AIR FORCE INSTITUTE OF TECHNOLOGY

Wright-Patterson Air Force Base, Ohio

1

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DISSERTATION

Presented to the Faculty of the School of Engineering

of the Air Force Institute of Technology

Air University

In Partial Fulfillment of the

Requirements for the Degree of

Doctor of Philosophy

Max A. Stafford, B.S., M.S.

Maj USAF

May 1985

Approved for public release; distribution unlimited

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PREFACE

Application of the semigroup theory of operators to infinite-dimensional systems is an exciting, recent development in control theory. In the mid-1960's control theorists were finding scores of new applications for the rapidly developing "modern control theory" of finite-dimensional systems. It is fair to say that this new theory was met with considerable resistance and skepticism; and, on the other hand, it was certainly overrated by some at the time. Today, two decades later, new applications of modern control theory are still being discovered. Indeed, its usefulness is now questioned by few, and techniques developed from finite-dimensional modern control theory are routinely taught in universities around the world. I believe semigroup theory to be a natural extension of this theory to infinite-dimensional systems. Furthermore, I am convinced that the rudiments of this theory will soon be standard fare for most graduate students of control theory.

There are two motivating forces favoring the development of this new theory. First, there are several important areas which are currently developing that give rise to infinite-dimensional models. Some specific topics that I am aware of are (1) temperature distribution in the space shuttle [illegible], (2) [illegible] space structure [illegible], (3) [illegible] and (4) the [illegible] of a charged particle [illegible] ([illegible]). [illegible],

[illegible]

sional modern control theory, intuition for what can be accomplished is already established for many control theorists and engineers.

Several individuals have been helpful in the research and writing of this dissertation: Dr. Ray Zazworsky of the Air Force Weapons Lab for sponsoring the research, and Drs. Dennis Quinn and Peter Maybeck (AFIT) for their many hours of reading barely legible notes, listening to semi-coherent progress reports, and most importantly, for providing ample, sound criticism of the rough draft.

A special thanks goes to my advisor, Dr. David Audley (AFIT). He kept the entire research phase interesting by (gently but firmly) pushing me into areas which were personally challenging.

Several of my fellow students have been helpful and supportive as well. Dr. Ronald Fuchs made many valuable suggestions. In particular, he recommended the use of partitioned matrices for the many matrix multiplications needed in Chapter IV. Lt Bryan Preppernau also helped by supplying encouragement and listening to many technical arguments in their formative stages.

Ms. Fonda Lilly graciously consented to type the equations and scientific symbols in this dissertation. In spite of many late hours and very tedious equations, she has done her usual outstanding, thoroughly professional job.

Finally, and most importantly, I would like to express appreciation to my family. No one could ask for more support or understanding than I have received from my wife, Sherry, and children, Scott and

Sunny. This dissertation is dedicated to you.

Max A. Stafford

December, 1984

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III. Modelling the Dynamic Behavior of Intense Charged Particle Beams

IV. Analysis of the Electrostatic Approximation Model

V. Summary and Suggested Areas for Further Research

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ABSTRACT

Dynamic models of a charged particle beam subject to external electromagnetic fields are cast into the abstract Cauchy problem form. Various applications of intense charged particle beams, i.e., beams whose self electromagnetic fields are significant, might require, or be enhanced by, the use of dynamic control constructed from suitably processed measurements of the state of the beam. This research provides a mathematical foundation for future engineering development of such dynamic control designs for such beams.

Beginning with the Vlasov equation, successively simpler models of intense beams are presented along with their corresponding assumptions. [illegible] abstract Cauchy problem form [illegible] of [illegible]

ANALYSIS OF THE DYNAMIC BEHAVIOR OF AN INTENSE CHARGED PARTICLE BEAM USING THE SEMIGROUP APPROACH

I. Introduction

Specific Area of Research

This investigation is concerned with the problem of controlling a physical system which is most naturally described by a set of partial differential equations (PDE). The many successes in the application of the "state variable" or "modern control theory" approach to systems of linear ordinary differential equations (ODE) have led many researchers to look for a practical extension of this theory to accommodate systems of linear PDE. As recently as 1978, however, a prominent researcher observed (Russell, 1978:640): "The control theory of partial differential equations has followed right on the heels of that for ordinary differential equations, but with slower and heavier tread."

Models for many physical systems can be brought into the form of an abstract Cauchy problem. Let X be a Banach space, and suppose A is a linear operator from a subset of X into X. If the domain of A is dense in X, then the equation

$$\dot{u}(t) = Au(t) \qquad t>0$$

and is denoted by δF_x. Existence of the Gateaux derivative of F at x does not imply continuity of the operator F.

On the other hand, one can generalize the derivative of an operator in a manner which mimics the "differentiability implies continuity" property of the usual derivative. Suppose for some $x \varepsilon X_0$, an open subset of $\mathcal{D}(F)$, there exists a $\hat{\delta} F_x \varepsilon B(X,Y)$ such that

$$\lim_{\|h\|_X \to 0} \frac{\left\| F(x+h) - F(x) - \hat{\delta} F_x(h) \right\|_Y}{\|h\|_X} = 0$$

The operator $\hat{\delta} F_x$ is termed the <u>Frechet derivative</u> of F at x, and the existence of this derivative implies continuity of F at x (Luenberger, 1969: 173). Furthermore, existence of $\hat{\delta} F_x$ implies existence of δF_x and the two are equal in this case.

The Gateaux and Frechet derivatives are often used to construct a linear approximation of a nonlinear operator. The procedure is analagous to the familiar first-order Taylor series linearization techniques for a real function of a real variable.

Consider next a function $u: I \to X$, where I is an interval (possibly infinite) of the real line, and X is a Banach space. If the Frechet derivative of u at $t_0 \varepsilon I$ exists, then this operator is called the <u>strong derivative</u> of u at t_0. For this special case, the cumbersome Frechet derivative notation $\hat{\delta} u_{t_0}$ is replaced with the usual differentiation symbols $\frac{d}{dt}u(t_0)$ or $\dot{u}(t_0)$.

$$D^k f = \frac{\partial^{|k|} f}{\partial x_1^{k_1} \partial x_2^{k_2} \ldots \partial x_n^{k_n}}$$

where $k=(k_1, k_2, \ldots k_n)$, the k_i being nonnegative integers, and,

$$|k| = \sum_{i=1}^{n} k_i$$

Unless stated otherwise, use of the usual differentiation symbols D_{x_k} or $\frac{\partial}{\partial x_k}$ indicates a generalized derivative. Various functional analysis texts cover generalized derivatives in detail (also known as distributional, and, more generally, as weak derivatives). See (Curtain and Pritchard, 1977: 136-138; Yosida, 1968: 48-52), for example.

Consider now an operator F (not necessaily linear) with domain $\mathcal{D}(F)$ a subspace of a normed linear space, X, and range contained in a normed linear space Y. Two generalizations of the derivative of a real function of a real variable are possible, where an appropriate topological generalization of R is assigned.

First, consider

$$\lim_{h \to 0} \frac{F(x+hv) - F(x)}{h}$$

where x, $v \in \mathcal{D}(F)$, and $h \in R$. If the limit exists for every $v \in \mathcal{D}(F)$, then the operator F is said to be Gateaux differentiable at x. In this case, the limit above defines a unique element in Y for every $v \in \mathcal{D}(F)$. This mapping is called the <u>Gateaux derivative</u> of F at x

denoted by $\mathbb{C}(X,Y)$, or by $\mathbb{C}(X)$ if $X=Y$. For examples and further discussions of closed operators the reader is referred to (Curtain and Pritchard, 1977: 45; Belleni-Morante, 1979: 60-63; Taylor and Lay, 1980: 208-217).

Derivatives. Various generalizations of the usual derivative of a real function of a real variable exist, depending on the topological properties assigned to the underlying spaces. Three specific types of derivatives are of use in the application of semigroup theory: (1) generalized derivatives, (2) Gateaux derivatives, and (3) Frechet derivatives. The Frechet derivative of a function whose domain is an interval of the real line is known as a strong derivative. Since the strong derivative is frequently used in semigroup theory, it is also discussed below.

Let $C_0^\infty(\Omega)$ denote the set of all functions ϕ which are continuous, have continuous partial derivatives of any order, and which have support bounded and contained in Ω, an open subset of R^n. The generalized derivative of any function $f \varepsilon L^1_{loc}(\Omega)$ (f only "locally" belongs to $L^1(\Omega)$, i.e., f is defined on Ω and is in $L^1(K)$ for every Lebesgue measurable set K whose closure is contained in Ω), if it exists, is defined to be the function g such that

$$\int_\Omega f(x) D^k \phi(x) dx = (-1)^k \int_\Omega g(x) \phi(x) dx$$

for all $\phi \varepsilon C_0^\infty(\Omega)$. The differentiation operator D^k is defined as the following two statements hold (Naylor and Sell, 1982: 240):

(i) If T is continuous at any point $x_0 \in X$, then it is continuous at every $x \in X$.

(ii) T is bounded if and only if it is continuous.

Although bounded linear operators are simpler to analyze, unbounded linear operators frequently appear in applications. The "derivative" operator, for example, is often unbounded, depending upon the domain and codomain chosen for a specific model. In some cases an unbounded linear operator enjoys properties similar to those of a continuous one, in which case it is termed a closed operator. The definition of a closed operator is often stated in terms of its graph, but an equivalent and more practical definition, in the context of metric spaces, is the following:

Definition 2.2 (Closed Operator)

Let $T:\mathcal{D}(T)\subset X\rightarrow Y$ be an operator with X, Y Banach spaces. Suppose $\{x_n\}$ is a sequence in $\mathcal{D}(T)$ with the properties

(i) $x_n \rightarrow x$

(ii) $Tx_n \rightarrow y$

The operator T is closed if $x \in \mathcal{D}(T)$ and $Tx=y$ for every such sequence in $\mathcal{D}(T)$.

The set of all closed linear operators defined on a subset of the Banach space X and with range contained in the Banach space Y is

X are especially useful in applications. The interval I typically represents values of time, while the Banach space X is most often a space of functions (e.g., $L^2(\Omega)$ or $H^1(\Omega)$). Suppose u denotes such a mapping (i.e., $u: I \subset R \to X$), then in accordance with Definition 2.1, u is continuous at $t_0 \in I$ if for every $\varepsilon > 0$ there exists a δ such that

$$\| u(t) - u(t_0) \|_X < \varepsilon$$

for all $t \in I$ satisfying

$$|t - t_0| < \delta$$

In semigroup theory, mappings from an interval I of the real line into the space $B(X)$ (see the previous subsection for the definition of $B(X)$) are of interest. If the general definition of continuity (Definition 2.1) [illegible] for a mapping $T: I \to B(X)$ at [illegible] t_0, [illegible] t_0 (see [illegible]). On the other hand, one could [illegible] in each [illegible]

$$[illegible]$$

[illegible]

Linear Operators on Banach Spaces. There are several key concepts involving linear operators whose domains and codomains are subsets of normed spaces. A bounded linear operator $T:X \to Y$, X, Y Banach spaces, is one for which a nonnegative real number K exists such that

$$\|Tf\|_Y \leqq K\|f\|_X$$

for all $f \varepsilon X$. The set of all bounded linear operators from X into Y is itself a Banach space and is denoted by $B(X,Y)$, or, if $X=Y$, by $B(X)$. The infimum of the set of all constants K satisfying the above inequality is the norm of T on the Banach space $B(X,Y)$ or $B(X)$.

Continuity. The notion of continuity of a function is fundamental in semigroup theory analysis. The following definition is sufficiently general for subsequent discussions:

Definition 2.1 (Continuity)

Let X , Y be normed linear spaces, and suppose F represents a function from a subset D of X into Y — i.e., $F:D \subset X \to Y$. F is said to be continuous at the point x_0 in D if for every real number $\varepsilon > 0$ there exists a real number δ such that

$$\|F(x) - F(x_0)\|_Y < \varepsilon$$

for all $x \varepsilon D$ satisfying

$$\|x - x_0\|_X < \delta$$

Mappings from an interval I on the real line into a Banach space

Various linear spaces are used in this work. The set of real numbers and the set of complex numbers are symbolized by R and C, respectively, and these are the only scalar fields used. The symbols R^n and C^n denote n-fold Cartesian products of the linear spaces R and C (with the usual addition and scalar multiplication definitions). The letters I and Ω are used to mean an interval of R or R^n, respectively, either finite or infinite — i.e., $I = (a,b) \subset R$, and $\Omega = \{x \in R^n : x = (x_1 \ldots x_n), a_i < x_i < b_i, i = 1 \ldots n\} \subset R^n$ with a, $a_i \in R$ or $-\infty$, and $b, b_i \in R$ or $+\infty$. Occasionally Cartesian products of linear spaces are denoted by the product symbol, Π. Specifically, letting $\{X_i\}_{i=1}^n$ be a set of linear spaces, the Cartesian product of these spaces is written as $\prod_{i=1}^n X_i$. The most common function spaces used in this report are the Lebesgue and Sobolev spaces, $L^p(\Omega)$ and $H^q(\Omega)$. The L^p spaces consist of (equivalence classes of) functions f such that $|f|^p$ is integrable in the Lebesgue sense. The Sobolev spaces, $H^q(\Omega)$, consist of the sets of functions f whose generalized derivatives (discussed below) up to and including order q are in $L^2(\Omega)$. For further discussion of Lebesgue and Sobolev spaces, the reader is referred to (Royden, 1968: Ch 6) and (Yosida, 1968: 55), respectively. The Sobolev spaces are Hilbert spaces for all integer $q \geq 0$, as is $L^2(\Omega)$, and $L^p(\Omega)$ is a Banach space for all integer $p \geq 1$. The norm of a function f in these normed spaces, or any other normed space, is symbolized by $\|f\|_X$ where X represents the space, or by $\|f\|$, if it is clear which space is intended.

II. Pertinent Results from Operator Semigroup Theory

Introduction

Some known results from the semigroup theory of operators (hereafter referred to as semigroup theory) are now presented. The theory has been rigorously developed by Hille and Phillips (1957). More recent texts have been written which emphasize the practical aspects of the theory (Butzer and Berens, 1967; Belleni-Morante, 1979; Curtain and Pritchard, 1977; Curtain and Pritchard, 1978; Davies, 1980; Fattorini, 1983; Pazy, 1983; Walker, 1980). The intent of this chapter is to state notation, definitions, and specific results pertaining to semigroup theory and the abstract Cauchy problem which are relevant to an analysis of the beam models developed in the following chapter.

Fundamental Notation and Definitions

Functions and Spaces. Let A, B be arbitrary sets. The notation $f: A \to B$ is used to denote a function f with domain $\mathcal{D}(f)$ equal to A, and range $\mathcal{R}(f)$ a subset of the codomain, B. The term operator is used to denote any function whose domain or codomain (or both) is a space of functions.

these readers should follow the remaining chapters with little difficulty.

Chapter III introduces various dynamic models of charged particle beams. The sophisticated "microscopic" models are presented first, and a linearization is performed in order to bring this class of models into the abstract Cauchy problem form. "Macroscopic" descriptions are then discussed in general, and a linear, single degree of freedom model is derived. Finally, a tractable model is developed in detail in order to illustrate semigroup theory techniques analytically.

An analytical solution of the "electrostatic approximation model" is thoroughly developed in Chapter IV. Various simplifying assumptions are introduced in Chapter III in the development of this model which would not necessarily be required if a numerical solution were sought. It is considered far more useful from a researcher's point of view, however, to develop a closed-form solution to the electrostatic approximation model thoroughly than to resort to a numerical solution of a more complicated model.

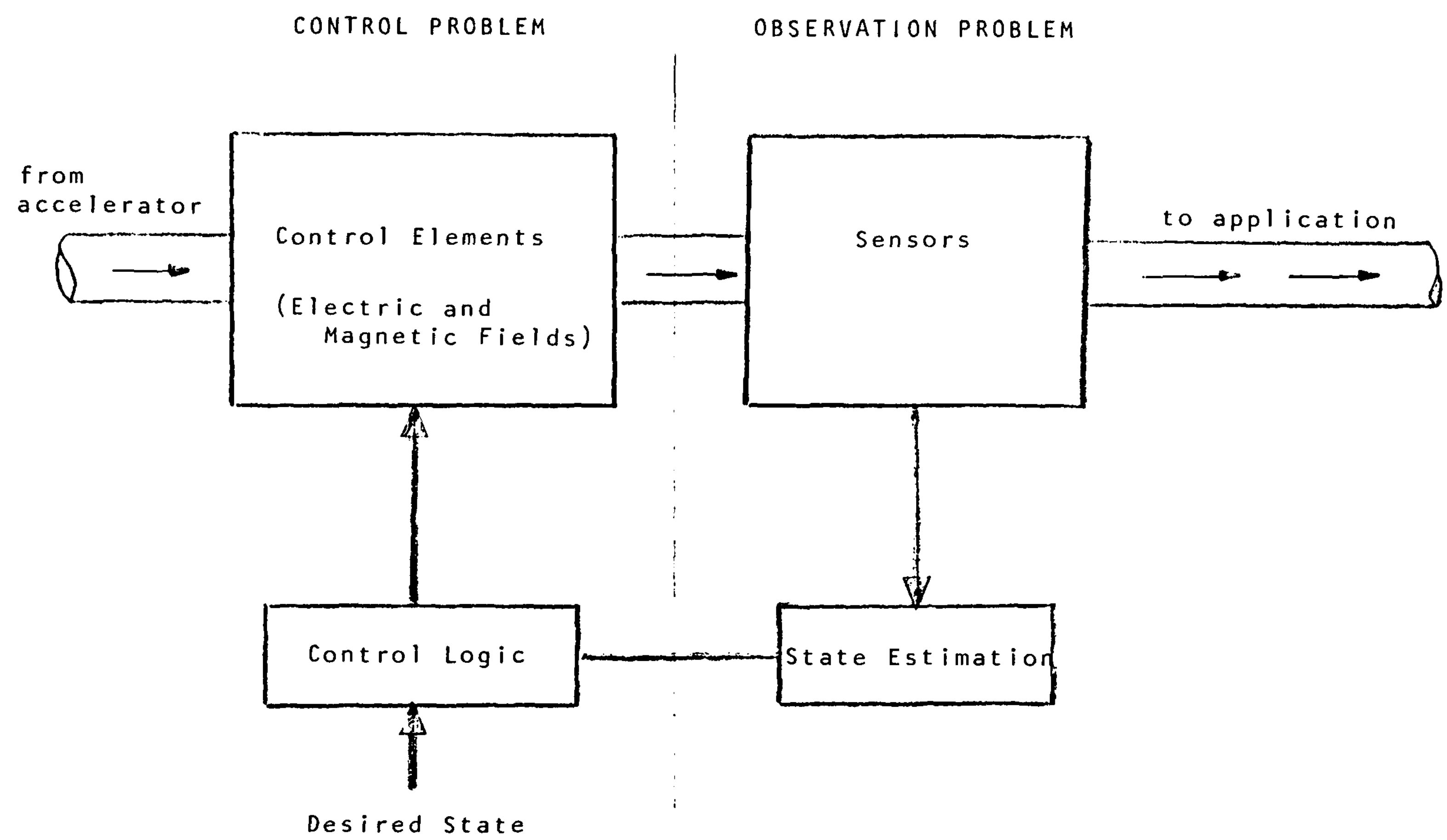
A summary of dissertation research results is presented in the concluding chapter, along with some suggested further areas of research.

nuclear fusion research has been treated in a manner similar to that herein (cf. Wang and Janos, 1970). The plasma confinement problem differs considerably from the beam dynamics problem, however. The plasma in Wang's work is assumed to be neutral, while a charged particle beam is a nonneutral plasma. Furthermore, the configuration of the plasma confinement problem does not at all match that of the beam problem, where a large velocity field in one direction is assumed. Nonetheless, the starting point for both plasma confinement in Wang's paper and the beam dynamics analysis in this dissertation is the Vlasov-Maxwell system of equations.

Overview

Three major topics are presented in the sequel: (1) a summary of relevant mathematical concepts, (2) a description of various mathematical models of the dynamics of a charged particle beam, and (3) an illustration of the theory.

The purpose of Chapter II is twofold. First, it provides readers with functional analysis and operator semigroup theory in their backgrounds a summary of notation, definitions, and results in these areas. Second, readers of this chapter with finite-dimensional modern control theory in their backgrounds are provided a glimpse of how the finite-dimensional theory generalizes to the infinite-dimensional theory. For example, a real matrix operator of order n is discussed as a special case in the subsection "Some Familiar Operators." Armed with these insights, and intuition provided by finite-dimensional theory,



THE BEAM DYNAMICS PROBLEM

primary goal of this dissertation is to advance the development of semigroup theory techniques by attacking a specific initial value problem: the dynamic behavior of an intense charged particle beam.

Intense beams of charged particles are beginning to be used in a wide variety of applications (Septier, 1983: xii). The dynamic behavior of such beams is quite complex because electromagnetic fields are affected by not only the positions of the particles, but by their velocities as well. Frequently the Vlasov-Maxwell system of PDE is chosen as a starting point for analysis of a collection of charged particles. Simplifying assumptions are often appropriate, but the resulting models are generally systems of PDE also. Analysis of the dynamic behavior of intense charged particle beams is an excellent choice, then, for an application of semigroup theory since (1) such beams are useful, and, (2) models of these beams are inherently distributed parameter systems of equations.

This dissertation establishes a framework for analyzing the <u>beam dynamics problem</u>. In the figure on page I-5 the basic problem is divided into two sub-problems: (1) the control problem, which is concerned with modifying the dynamic behavior to achieve some desired state, and (2) the observation problem, which is concerned with determining the present state of the beam. The foundation laid in this work is original and should serve to direct and organize beam dyamics research in the future.

Some articles exist in the literature which are related to this research. For example, the plasma confinement problem associated with

$$A = \sum_{k=1}^{m} A_k(x)\frac{\partial}{\partial x_k} + B(x)$$

and where $A_k(x) = \{\alpha_{ij}^k(x)\}$ and $B(x) = \{\beta_{ij}(x)\}$ are n^{th}-order matrix functions defined in R^m.

It has been shown that ordinary, partial, stochastic, and delay differential equations can all be accommodated by the application of semigroup theory to initial value problems on a Banach space (Curtain and Pritchard, 1978: Ch 8). Belleni-Morante (Belleni-Morante, 1979: Ch 8-13) discusses in detail the following specific problems: heat conduction in rigid bodies, one-speed neutron transport, kinetic theory of vehicular traffic, the telegraphic and wave equations, the one-dimensional Schrödinger equation, and stochastic population theory. Additionally, Markov processes were studied from the semigroup theory point of view by Hille, Yosida and Feller in the early 1950's (Fattorini, 1983: 98). These examples, and many others that can be found in the recent literature, illustrate the wide variety of physical problems that can be formulated and analyzed within the context of semigroup theory.

This diversity of applications is encouraging, but far more practical applications are needed. Fattorini (Fattorini, 1983: xx) states, "Nowadays, many volumes devoted ... to the treatment of semigroup theory exist... In contrast, accounts of the applications to particular partial differential equations ... are scarcer..." This suggests that more applications should be attempted in order for the theory to develop into a practical, working body of knowledge. The along with an initial condition,

$$u(0) = u^0$$

is termed an abstract Cauchy problem.

Analysis of the abstract Cauchy problem can be performed with the semigroup theory of operators. This approach has many parallels with the modern control theory approach to systems of linear, time invariant, first-order ordinary differential equations:

$$\dot{x}(t) = Ax(t) \qquad t \geq 0$$

where A is an n by n real matrix. For example, the state transition matrix, e^{At}, for such a system of ODE, is an element of a semigroup of operators $\{e^{At}\}$ generated by A, where $t \geq 0$. Another parallel exists in that the semigroup theory emphasizes spectral properties of the operator A in the abstract Cauchy problem. This is, of course, analogous to the modern control theory emphasis on the eigenvalues and eigenvectors of the matrix A. These parallels provide a compelling case for considering an appropriate extension of modern control theory to be analysis of the abstract Cauchy problem through the semigroup theory of operators. This point of view is adopted in the present work.

Only linear systems of partial differential equations are considered herein. In fact, all models are of the form

$$\frac{\partial}{\partial t} w(x,t) = Aw(x,t)$$

where $w(x,t) \varepsilon R^n$, $x=(x_1 \ldots x_m) \varepsilon R^m$, A is given by

Spectral Analysis Definitions. It is well known that the eigenvalues and eigenvectors in a finite-dimensional system of linear, first-order, time-invariant, differential equations are instrumental in an analysis of such a system. In infinite-dimensional systems the eigenstructure is equally important.

Let $A:\mathcal{D}(A)\rightarrow X$, $\mathcal{D}(A)\subset X$, be a linear operator with X a Banach space. The set of all complex numbers can be partitioned into two subsets according to whether $\lambda I-A$ satisfies the following three conditions for $\lambda\varepsilon C$ (Yosida, 1968: 209; Curtain and Pritchard, 1977: 163, 164; Naylor and Sell, 1982: 414-429):

(i) $(\lambda I-A)^{-1}$ exists

(ii) $(\lambda I-A)^{-1}$ is continuous

(iii) the range of $\lambda I-A$ is dense in X

The set of all $\lambda\varepsilon C$ such that these conditions are met is called the resolvent set, while the set of all other complex numbers is called the spectrum. The resolvent set and spectrum are denoted by $\rho(A)$ and $\sigma(A)$, respectively.

The spectrum of a linear operator A defined on a subset of a finite-dimensional space E, with range contained in E, consists of only those $\lambda\varepsilon C$ such that $\lambda I-A$ is not injective. Similarly, $\lambda I-A$ can fail to be injective for some values of λ, but, unlike the finite-dimensional case, the spectrum may contain other complex numbers. In fact, there are three disjoint subsets of $\sigma(A)$. The point spectrum consists of those $\lambda\varepsilon C$ for which $\lambda I-A$ is not injective. The continuous spectrum is made up of those λ for which

$(\lambda I-A)^{-1}$ exists but is not continuous, and, for which the range of $\lambda I-A$ is dense in X. Finally, the <u>residual spectrum</u> consists of those λ for which $(\lambda I-A)^{-1}$ exists and is continuous, but such that the range of $\lambda I-A$ is not dense in X.

Let the notation $R(z,A)$ denote the operator $(zI-A)^{-1}$ for any $z\varepsilon\rho(A)$. The following two facts are established in (Belleni-Morante, 1979: 62,63):

(i) If A is a closed linear operator $(A\varepsilon\mathfrak{C}(X))$, A_b is a bounded linear operator $(A_b\varepsilon B(X))$, and if the domain of A_b contains the domain of A, then $A+A_b\varepsilon\mathfrak{C}(X)$.

(ii) If for any $z_0\varepsilon C$, $R(z_0,A)\varepsilon B(X)$, then $A\varepsilon\mathfrak{C}(X)$.

These two facts are used frequently in practical applications of the semigroup theory of operators. For example, see (Belleni-Morante, 1979: 179) where the first fact is used in proving an important perturbation theorem.

The set of closed linear operators is frequently partitioned in a manner which simplifies semigroup theory discussions. The four classes of interest are denoted by $\mathcal{G}(1,\beta)$, $\mathcal{G}'(1,\beta)$, $\mathcal{G}(M,\beta)$, and $\mathcal{G}'(M,\beta)$ and are defined as follows (Belleni-Morante, 1979:140,141,145):

<u>Definition 2.3</u> (<u>$\mathcal{G}$-Classes</u>)

Let $A\varepsilon\mathfrak{C}(X)$, $\mathcal{D}(A)$ dense in X, $z\varepsilon C$, and $\zeta=\mathrm{Re}(z)$. Then A is in the class

(i) $\mathcal{G}(1,\beta)$ if $\{z:\zeta>\beta\}\subset\rho(A)$ and $\|R(z,A)\|\leq\frac{1}{\zeta-\beta}$ for all z such that $\zeta>\beta$

(ii) $\mathcal{G}'(1,\beta)$ if $\{z:|\zeta|>\beta\}\subset\rho(A)$ and $\|R(z,A)\|\leq\frac{1}{|\zeta|-\beta}$

for all z such that $|\zeta|>\beta$

(iii) $\mathfrak{G}(M,\beta)$ if $\{z:\zeta>\beta\}\subset\rho(A)$ and for any integer $j=1,2,...$

$$\left\|R(z,A)^j\right\| \leqq \frac{M}{(\zeta-\beta)^j}$$

for all z such that $\zeta>\beta$

(iv) $\mathfrak{G}'(M,\beta)$ if $\{z:|\zeta|>\beta\}\subset\rho(A)$ and for any integer $j=1,2,...$

$$\left\|R(z,A)^j\right\| \leqq \frac{M}{(|\zeta|-\beta)^j}$$

for all z such that $|\zeta|>\beta$

The various mathematical symbols which have appeared in this section are summarized in Appendix A. With these fundamental definitions and results in mind, attention is now turned to the abstract Cauchy problem.

The Abstract Cauchy Problem

Mathematical models are frequently developed to predict the dynamic behavior of certain variables in a physical system. In many cases the model is finite-dimensional and one is interested in knowing what values in R each variable assumes at any given time. In distributed systems, however, the variables of interest can be elements of a function space at each instant of time. Quite often a mathematical model of a physical system, whether finite or infinite-dimensional, can be expressed as an abstract Cauchy problem.

Definition 2.4 (Abstract Cauchy Problem)

Let the linear operator $A:\mathcal{D}(A)\to X$ have domain dense in the Banach space X. The abstract Cauchy problem consists of finding a solution to the differential equation and initial condition

$$\frac{d}{dt}\,u(t) = Au(t) \qquad (t>0) \tag{2.1}$$

$$u(0) = u^0 \qquad u^0 \in X \tag{2.2}$$

where $\frac{d}{dt}\,u(t)$ denotes the strong derivative.

Definition 2.5 (Solution)

A solution of the abstract Cauchy problem (2.1), (2.2) is any continuous function $u:[0,\infty)\to X$ which

(i) is continuously differentiable at every $t>0$

(ii) is an element of $\mathcal{D}(A)$ for every $t>0$, and

(iii) satisfies equation (2.2).

In applications, there are usually further mathematical requirements that must be met, rather than simply the existence of a solution for a single initial condition. The following definition is crucial to the development of useful solutions to the abstract Cauchy problem (see Fattorini, 1983: 29,30):

Definition 2.6 (Well Posed)

The abstract Cauchy problem (2.1), (2.2) is well posed in if the following two conditions are satisfied:

(i) Existence of solutions for sufficiently many initial data: There exists a dense subspace D of X such that, for any $u^0 \varepsilon D$, there exists a solution of the abstract Cauchy problem.

(ii) Continuous dependence of solutions on their initial data: There exists a nondecreasing, nonnegative function $C(t)$ defined in $t \geqq 0$ such that

$$\left\| u(t) \right\| \leqq C(t) \left\| u(0) \right\| \qquad (2.3)$$

for any solution of the abstract Cauchy problem.

These requirements are similar to those generally deemed essential in order for a mathematical model to correspond to physical reality (e.g., see *Courant and Hilbert, 1962: 227*): (1) existence of solutions, (2) uniqueness of solutions, and (3) continuous dependence of the solution on the initial data. For instance, the well posed Cauchy problem has the existence-of-solution property for a particular set of initial conditions. (However, a solution is not always guaranteed to exist for every $u^0 \varepsilon X$, but only for every u^0 in a dense subset of X.) Furthermore, equation (2.3) ensures that any solution of a well posed abstract Cauchy problem is unique. To demonstrate this, let v, w be solutions of

$$\frac{d}{dt} u(t) = Au(t) \qquad (t>0)$$

$$u(0) = u^0 \qquad u^0 \varepsilon X$$

and consider the vector $v-w$. Clearly, $v-w$ is also a solution, and, since $(v-w)(0)=0$, we have from equation (2.3)

$$\|(v-w)(t)\| \leq C(t)\|0\| = 0$$

from which it follows that $v=w$. Heuristically, the third requirement is that two initial conditions which are "close" to each other should yield solutions which are also "close." The second posedness condition ensures this <u>continuous dependence of solutions on the initial data</u>.

Any solution of a well posed abstract Cauchy problem with initial condition lying in D (the set referred to in the first posedness condition) uniquely defines an operator $S(t):D \to X$ as

$$u(t) = S(t)u^0$$

for $t \geq 0$, with $u(0) = u^0$. Furthermore, $S(t)$ is necessarily a linear, bounded operator in D (by the linearity of A in equation (2.1), and by the second posedness condition, respectively) and, as D is dense in X , $S(t)$ can be extended to all of X . The operator-valued function S is called the <u>propagator</u> for the solution of the well posed abstract Cauchy problem.

Well posedness of the abstract Cauchy problem supports a notion of "solution" for any $u^0 \varepsilon X$. Indeed, suppose the sequence $\{u_n\}_1^\infty \subset D$ is such that $u_n \to u^0$. Well posedness provides that the functions $S(\cdot)u_n \varepsilon C([0,\infty);X) \cap C^1([0,\infty);X)$ converge uniformly to $S(\cdot)u^0 \varepsilon C([0,\infty);X)$ which may not be a solution in the sense of Definition 2.5, but which will be called a <u>generalized solution</u> if

$u^0 \varepsilon D$ (this is the same as the usual notion of a weak solution — see (Fattorini, 1983: 30,31)).

It is difficult, in general, to determine whether an abstract Cauchy problem is well posed and, hence, whether there exists a propagator for an arbitrary mathematical model with the form of equations (2.1), (2.2). If the linear operator A in equation (2.1) satisfies certain conditions, however, the propagator can be shown to exist and, iterative schemes are known for its construction. Specifically, if $A \varepsilon B(X)$, or if A is in any of the $\mathfrak{G}$-classes defined previously, then the propagator exists and can be constructed by an iterative process. The details of this assertion are now presented.

presented.

Semigroups, Groups, and Solutions

A collection of theorems concerning solutions of the abstract Cauchy problem is now given. These theorems are presented in order to lay a foundation for discussions in Chapters IV and V.

The words semigroup and group are defined in most modern algebra texts (for example, see (Birkhoff and Bartee, 1970: 197-204)). Let S represent a set, and suppose a function exists with domain $S \times S$ and codomain S. Any such function is called a binary operation. A set S along with a specific binary operation $*$ is termed a binary algebra. If the binary operation in a binary algebra is endowed with the associative property, then it is called a semigroup. Furthermore, a semigroup with an identity element is [illegible].

Some brief examples of [illegible]. Let S_1 represent the set of positive [illegible]; i.e., $S_1 = \{i : i = 2, 3, \ldots\}$. [illegible] defined [illegible]

[illegible]

[illegible]

$$\alpha_1(\alpha_1(j,k),\ell) = (j^k)^\ell \neq j^{(k^\ell)} = \alpha_1(j,\alpha_1(k,\ell))$$

The same set S_1 with a different binary operation α_2 can form a binary algebra that is a semigroup, however. Let α_2 be defined by

$$\alpha_2(j,k) = j \cdot k$$

where "$\cdot$" represents ordinary multiplication of real numbers. Now the binary algebra (S_1,α_2) possesses the associative property since

$$\alpha_2(\alpha_2(j,k),\ell) = (j \cdot k) \cdot \ell = j \cdot (k \cdot \ell) = \alpha_2(j,\alpha_2(k,\ell))$$

Note that (S_1,α_2) is not a monoid because no element n in S_1 can satisfy the requirements of an identity:

$$\alpha_2(n,j) = \alpha(j,n) = j \qquad n,j \in S$$

By altering the set S_1 slightly, however, a monoid can be generated. Let S [illegible] be the set [illegible] all positive integers, i.e., [illegible] $\{1,2,\ldots\}$ [illegible] since (S, α) is a semigroup which [illegible]

[illegible]

whose every element is invertible. (S_2, α_2) introduced above is not a group since only the identity element of S_2 is invertible. By enlarging S_2 to include all rational numbers greater that zero, a group can be constructed. For the set $S_3 = \{r : r = p/q, p, q = 1, 2, \ldots\}$ it is straightforward to show that (S_3, α_2) is a group.

Consider now the set $\mathcal{S} = \{S(t) : t \geq 0\}$, where S is the propagator of a well posed abstract Cauchy problem. Let a binary operation α be defined on $\mathcal{S} \times \mathcal{S}$ by

$$\alpha(S(t_1), S(t_2)) = S(t_1) \circ S(t_2) \qquad (t_1, t_2 \geq 0)$$

where the symbol "$\circ$" represents the composition of two functions. The following theorem summarizes several important properties of the propagator S (Fattorini, 1983: 63):

<u>Theorem 2.1</u>

If $\mathcal{S} = \{S(t) : t \geq 0\}$, where S is the propagator for a well posed abstract Cauchy problem, and α is the binary operation defined above, then

(i) $(\mathcal{S}, \alpha)$ is a monoid

(ii) for $t_1, t_2 \geq 0$, $S(t_1 + t_2) = S(t_1) \circ S(t_2)$

(iii) the operator $S : [0, \infty) \to \mathcal{B}(X)$ is strongly continuous at every $t > 0$, and strongly continuous from the right at $t = 0$.

(iv) there exist nonnegative constants M, β such that

$$\|S(t)\| \leq Me^{\beta t}$$

The term <u>strongly continuous group</u> is defined in a similar fashion. Consider a set $\mathcal{S}' = \{S(t) : t \in R\}$, and let α' be the binary operation defined by

$$\alpha'(S(t_1),S(t_2)) = S(t_1)\circ S(t_2)$$

If this composition satisfies

$$S(t_1+t_2) = S(t_1)\circ S(t_2)$$

for all $t_1, t_2 \varepsilon R$, and if S is strongly continuous at $t=0$, then S is referred to as a <u>strongly continuous group</u> (Curtain and Pritchard, 1977: 149; Fattorini, 1983: 81).

The following theorem is the most important result in the study of the abstract Cauchy problem. It provides necessary and sufficient conditions, in terms of the operator A and its resolvent $R(z,A)$, for the abstract Cauchy problem to be well posed (Fattorini, 1983: 65).

<u>Theorem 2.2</u>

Let the operator A in equation (2.1) be closed. The abstract Cauchy problem (2.1), (2.2) is well posed and its propagator S satisfies

$$\|S(t)\| \leq Me^{\beta t} \qquad (t \geq 0)$$

if and only if $A \varepsilon \mathfrak{G}(M,\beta)$.

A similar result for the abstract Cauchy problem on the whole real line exists (Fattorini, 1983: 72):

Theorem 2.3

Let the operator A in the abstract Cauchy problem

$$\dot{u}(t) = Au(t) \qquad -\infty<t<\infty \qquad (2.4)$$

$$u(0) = u^0 \qquad u^0 \varepsilon \mathcal{D}(A) \qquad (2.5)$$

be closed. This Cauchy problem is well posed and its propagator S satisfies

$$\|S(t)\| \leqq Me^{\beta|t|} \qquad -\infty<t<\infty$$

if and only if $A \varepsilon \mathfrak{G}'(M,\beta)$.

It is useful now to state a definition and some results from semigroup theory. Theorems 2.2 and 2.3 are usually difficult to apply directly, but the results below improve the situation somewhat.

Definition 2.7 (Infinitesimal Generator) (Curtain and Pritchard, 1977: 150,151; Fattorini, 1983: 81)

Let $\mathcal{S} = \{S(t):t\geqq 0\} \subset B(X)$ be a strongly continuous semigroup. The operator A defined by

$$Au = \lim_{t\to 0+} \frac{S(t)u-u}{t}$$

whenever the limit exists, is the infinitesimal generator of $\mathcal{S}$.

The phrase "A generates a strongly continuous semigroup $\mathcal{S}$" is frequently used to mean that A is the infinitesimal generator of $\mathcal{S}$. The following two theorems are proven in (Fattorini, 1983: 81-83):

Theorem 2.4

The linear operator A generates a strongly continuous semigroup $\{S(t):t\geq 0\}$, with the property

$$\|S(t)\| \leq Me^{\beta t}$$

if and only if $A\varepsilon\mathfrak{G}(M,\beta)$.

Theorem 2.5

The linear operator A generates a strongly continuous group $\{S(t):-\infty<t<\infty\}$, with the property

$$\|S(t)\| \leq Me^{\beta|t|}$$

if and only if $A\varepsilon\mathfrak{G}'(M,\beta)$.

Summarizing the results thus far, it is apparent that the problem of showing the operator A in equation (2.1) (equation (2.4)) to be an element of $\mathfrak{G}(M,\beta)$ ($\mathfrak{G}'(M,\beta)$) is equivalent to showing the (corresponding) abstract Cauchy problem to be well posed. In order to go further and actually solve a well posed Cauchy problem, one needs to construct the semigroup generated by A since this semigroup is the propagator for the problem and provides the solution $u(t) = S(t)u^0$ of equations (2.4) and (2.5). Several special cases are now considered for the operator A in equation (2.4).

$A\varepsilon B(X)$. Construction of the semigroup operator is most easily accomplished when A in equation (2.4) is an element of $B(X)$. (It can be shown that $A\varepsilon B(X)$ implies that $A\varepsilon\mathfrak{G}'(M,\beta)$.) The following result follows directly from a theorem in (Belleni-Morante, 1979:

131):

<u>Theorem 2.6</u>

If $A \varepsilon B(X)$, then A generates the strongly continuous group $\{S(t): -\infty < t < \infty\}$ with $S(t)$ defined by

$$S(t) = \lim_{n\to\infty} \sum_{j=0}^{n} \frac{t^j A^j}{j!}$$

It can also be shown (Belleni-Morante, 1979: 130-133) that satisfies

$$\|S(t)\| \leqq e^{\|A\| |t|} \qquad (-\infty < t < \infty)$$

In light of Theorem 2.5 and the foregoing, it is clear that the solution of the abstract Cauchy problem (2.4), (2.5) is given by

$$u(t) = S(t)u^0$$

for any $u^0 \varepsilon X$.

<u>$A \varepsilon \mathfrak{G}(1,0)$, $A \varepsilon \mathfrak{G}'(1,0)$</u>. Consider next the case where $A \varepsilon \mathfrak{G}(1,0)$ in the Cauchy problem of equations (2.1), (2.2). Define a sequence of operators, $\{S_n(t)\}_{n=1}^{\infty}$ by

$$S_n(t) = \left[(I - \frac{t}{n}A)^{-1}\right]^n \qquad (t \geqq 0, \quad n = 1, 2, \ldots)$$

$$\underline{P}(\underline{x},t) = \frac{1}{n(\underline{x},t)} \int_{R^3} \underline{p} f(\underline{x},\underline{p},t) d^3p \tag{3.3}$$

$$\underline{V}(\underline{x},t) = \frac{1}{n(\underline{x},t)} \int_{R^3} \underline{v}(\underline{p}) f(\underline{x},\underline{p},t) d^3p \tag{3.4}$$

where $\underline{v}(\underline{p})=\underline{p}/\gamma m$. The <u>current density vector</u>, $\underline{J}(\underline{x},t)$, is given by

$$\underline{J}(\underline{x},t) = q\underline{V}(\underline{x},t)$$

Finally, the <u>pressure tensor</u>, $\mathcal{P}$, is defined as follows:

$$\mathcal{P}(\underline{x},t) = \int_{R^3} [\underline{p}-\underline{P}(\underline{x},t)][\underline{v}(\underline{p})-\underline{V}(\underline{x},t)]^T f(\underline{x},\underline{p},t) d^3p \tag{3.5}$$

The preceding definitions can all be expressed rigorously within the context of probability theory. Let $\Omega=R^6$, and denote the <u>Borel field</u> (Maybeck, 1979: 62) associated with R^6 by F. Define next a set function p_t for every $t\varepsilon[0,T]$ by

$$p_t(\{(\underline{x},\underline{p}):(\underline{x},\underline{p})\varepsilon B\}) = \frac{1}{N}\int_B f(\underline{x},\underline{p},t) d^3x d^3p$$

where $B\varepsilon F$, and f is a distribution function as defined above. For each $t\varepsilon[0,T]$, the triplet (Ω,F,p_t) forms a <u>probability space</u> (Maybeck, 1979: 64). Defining a new function f^* by

$$f^*(\underline{x},\underline{p},t) = \frac{1}{N}f(\underline{x},\underline{p},t)$$

effects. This approach becomes unwieldy for very large numbers of particles, but it is sometimes taken (Cohen and Killeen, 1983: 59). Generally, however, only macroscopic quantities are of interest, as opposed to the specific path of any single particle. Consequently, models of a plasma usually incorporate probability concepts.

The kinetic theory of plasmas is frequently developed by use of a <u>distribution function</u>* (Davidson, 1974: 11, 12; Reif, 1965: 494, 495; Krall and Trivelpiece, 1973: 5,6; Chen, 1974: 199, 200). Suppose there exists a collection of N charged particles and a function, $f: R^5 x[0,T] \rightarrow [0,\infty)$. f is called a distribution function if the product $f(\underline{x},\underline{p},t)d^3xd^3p$ yields the mean number of particles in the hypercube d^3xd^3p centered at $(\underline{x},\underline{p})$ at time t .
By integrating out the dependence of f on the momentum coordinates, the <u>number density</u>, $n(\underline{x},t)$, is obtained:

$$n(\underline{x},t) = \int_{R^3} f(\underline{x},\underline{p},t)d^3p \tag{3.2}$$

If the particles each have charge q, then the <u>charge density</u>, $\sigma(\underline{x},t)$, is given by

$$\sigma(\underline{x},t) = qn(\underline{x},t)$$

The <u>macroscopic momentum vector</u>, $\underline{P}(\underline{x},t)$, and the <u>macroscopic velocity vector</u>, $\underline{V}(\underline{x},t)$, are defined as follows:

*The reader is cautioned that the phrase "distribution function" in plasma physics literature is <u>not</u> synonymous with a "cumulative distribution function" in probability theory.

Notation and Definitions

Most of the notation in this chapter corresponds to that commonly found in plasma physics texts. A summary is given in Appendix B, but for the reader who is unfamiliar with this area, a discussion of some of the pertinent notation and definitions is now given.

Single Particle. Consider a particle of mass m and charge q in the presence of an electric field $\underline{E}$ and magnetic field $\underline{B}$. The force on the particle exerted by these fields is given by

$$\underline{F}(t) = q[\underline{E}(\underline{x},t) + \underline{v}(t)\times\underline{B}(\underline{x},t)]$$

where $\underline{v}(t)$ denotes the velocity vector of the particle. The relativistic version of Newton's second law of motion is

$$\frac{d}{dt}\,\underline{p}(t) = \underline{F}(t)$$

where $\underline{p}(t)$ is the mechanical momentum vector of the particle. The momentum vector is related to $\underline{v}(t)$ by

$$\underline{p}(t) = \gamma m\underline{v}(t) = (1-\beta^2)^{-\frac{1}{2}} m\underline{v}(t) \qquad (3.1)$$

where $\beta=|\underline{\beta}|=|\underline{v}(t)|/c$, and c is the vacuum speed of light.

Plasma. Now consider a collection of charged particles. It is always possible to write the equations of motion for each individual particle including inter-particle forces as well as external field to the summary of notation given in Appendix B in lieu of reading the following section.

The most complete description of a collisionless plasma consists of the self-consistent Vlasov and Maxwell equations. Models developed directly from these are known as microscopic descriptions (Davidson, 1974: 10). These equations are presented and a linear perturbation model is developed. This model is then shown to have the structure of an abstract Cauchy problem.

By "taking moments" of the Vlasov equation one can develop a chain of equations which are commonly referred to as macroscopic descriptions (Davidson, 1974: 14). The continuity and momentum equations are the first and second set of equations in the chain, and these are presented following the microscopic model discussions.

The microscopic and macroscopic descriptions are stated in Cartesian coordinates for ease of exposition, but typically a cylindrical coordinate system is more practical when invoking symmetry conditions. Therefore, a coordinate transformation is performed on the macroscopic equations. This facilitates development of a particular single degree of freedom nonlinear model. This model is linearized about an appropriately chosen equilibrium, and the resulting linear model is also shown to have the abstract Cauchy problem structure.

The final section is concerned with a further simplification of this single degree of freedom model which isolates certain dominant dynamic behavior.

of models with the abstract Cauchy problem form. Successful development of such models would invite application of the growing body of infinite dimensional modern control theory to particle beam dynamics problems.

The inter-particle forces are typically classified as either collective or collisional forces (Lawson, 1983: 2). Collective forces are those which depend only upon an average of the fields of many neighboring particles. Collisional forces, on the other hand, depend upon the detailed structure of the charge distribution. The models developed in this chapter deal only with the case in which collective forces dominate. Collisional forces are not considered since particle accelerators are generally designed to have low collisional frequencies.

The term "plasma" has been defined in various ways in the literature (Lawson, 1977: 3). In the present work, any collection of charged particles whose collective forces are not negligible, when compared with forces exerted by external fields, is termed a plasma.

In many applications particle beams are produced and transported some distance in a vacuum. All models in this chapter are developed under this assumption. Consequently, the assumptions made thus far can be simply stated as follows: this chapter is devoted to the presentation and development of dynamic models of a collisionless non-neutral plasma in a vacuum.

Overview. Notation and definitions from electrodynamics and plasma physics are stated first. Since the notation is essentially standard, the reader familiar with these two areas may wish to refer

III. Modelling the Dynamic Behavior of Intense Charged Particle Beams

Introduction

Problem Description and General Assumptions. In this research charged particle beam is considered to be any collection of charged particles having gross motion approximately parallel to some curve. The curve is called the axis and, in general, the cross-sectional shape of the beam varies along the axis. A wide range in complexity of beam models exists due to the fact that the particles are charged.

For a sufficiently low number density, the trajectories of charged particles are unaffected by the presence of other particles around them. In this case, the modelling process is relatively straightforward since overall beam behavior can be inferred from motions of individual particles. The study of trajectories of particles in low density particle beams is referred to as "charged-particle optics" (Lawson, 1977: 3) and is not considered here.

Inter-particle forces cannot be ignored at high number densities; far more complex and interesting models are required in this case. Most often these models consist of partial differential equations. Consequently, the study of the dynamic behavior of beams whose inter-particle forces cannot be neglected is a ripe area for the development devoted to presenting (and, in some cases, developing) some of these models.

The spectrum of A can be shown to be the empty set in this case.

<u>The Convection Operator</u> (Belleni-Morante, 1979: 340-344). Consider next the operator $A:\mathcal{D}(A)\rightarrow X$ defined by

$$Af = -v\frac{d}{dx}f$$

with $X=L^2(-\infty,\infty)$ and $\mathcal{D}(A)=\{f\varepsilon X:\frac{d}{dx}f\varepsilon L^2(-\infty,\infty)\}$. It can be shown that $A\varepsilon\mathcal{G}'(1,0)$ and, further, that the strongly continuous group $\{S(t):-\infty<t<\infty\}$ generated by A is characterized by

$$S(t)u^0(x) = u^0(x - vt)$$

Summary

This chapter has provided a necessary frame of reference for the next three chapters. Some notation and fundamental definitions were presented first, along with several references. Next the structure and some key concepts associated with the abstract Cauchy problem were introduced. The link between operator semigroup theory and the Cauchy problem was then established, along with several important results. Finally, some familiar operators were covered in the semigroup theory setting.

As indicated in the first chapter, a wide variety of partial differential equation models have been established to describe the dynamic behavior of a beam of charged particles. The next chapter is illustrate the wide applicability of the theory.

n^{th}-order Matrix. Let $A = A$, an n^{th}-order matrix of real numbers with $\mathcal{D}(A) = R^n = X$. In this case, the associated abstract Cauchy problem

$$\frac{d}{dt}x(t) = Ax(t)$$

$$x(0) = x^0$$

with $x = (x_1, x_2, \ldots x_n)$, is a finite-dimensional model ($\dim(X) = n$), and $A \varepsilon B(X)$. Consequently, by Theorem 2.6, A generates the strongly continuous group $\{S(t): -\infty < t < \infty\}$ where

$$S(t) = \lim_{n\to\infty} \sum_{j=0}^{n} \frac{t^j A^j}{j!} \equiv e^{At}$$

The spectrum of A consists of the n (or fewer) complex numbers λ for which

$$\det(\lambda I - A) = 0$$

An Integral Operator (Belleni-Morante, 1979: 136-138). Let the operator A be defined by

$$Af = \int_0^1 (x-y)f(y)dy$$

for every $f \varepsilon X = C[0,1]$. It is not difficult to show that $A \varepsilon B(X)$ and that $\|A\| \leq 1$. The strongly continuous group of operators $\{S(t): -\infty < t < \infty\}$ is defined by

$$S(t)f = f + \sqrt{12}\,\sin\left(\frac{t}{\sqrt{12}}\right)Af + 12\left(1 - \cos\frac{t}{\sqrt{12}}\right)A^2 f$$

The operator A is linear and has domain, $\mathcal{D}(A)$, in the Banach space X. The function g takes on a value in X for each $t \geq 0$. The following theorem provides sufficient conditions for this problem to have a unique solution (Fattorini, 1983: 87):

Theorem 2.12

Let the operator A in equation (2.6) be an element of the class $\mathcal{G}(M,\beta)$. If g is a continuously differentiable function on the interval $[0,T]$, then the unique solution of equations (2.6), (2.7) is given by

$$u(t) = S(t)u^0 + \int_0^t S(t-s)g(s)ds \qquad (0 \leq t \leq T)$$

where $\{S(t): t \geq 0\}$ is the strongly continuous semigroup generated by A.

Some Familiar Operators

Various operators which are familiar to engineers and physicists have been analyzed in the literature from the semigroup operator point of view. Results are now given for the following operators: (1) an n^{th}-order matrix of real numbers, (2) a specific integral operator, and (3) the (scalar) convection operator. The first example involves a bounded operator defined on a finite dimensional space, the second a bounded operator defined on an infinite dimensional space, and the final example deals with an unbounded operator defined on an infinite dimensional space. These specific operators are chosen solely to following result is very practical (Belleni-Morante, 1979: 179-182):

Theorem 2.11

If $A = A_b + A_u$, $A_b \varepsilon B(X)$, and $A_u \varepsilon \mathcal{G}(M,\beta)$, then $A \varepsilon \mathcal{G}(M, \beta + M \| A_b \|)$.

As a result of this theorem, one need "worry" only about the "unbounded portion" of an operator, usually the derivative terms. The strongly continuous semigroup for the operator A satisfying this theorem is constructed by an iterative process. Let $\{S(t): t \geq 0\}$ be the semigroup generated by A_u, and define the sequence $\{Z_j(t)\}_{j=1}^{\infty}$ by

$$Z_0(t)f = S(t)f \qquad (t \geq 0)$$

$$Z_{n+1}(t)f = S(t)f + \int_0^t S(t-s) A_b Z_n(s) f(s) ds \qquad (t \geq 0, \ n=1,2,\ldots)$$

The strongly continuous semigroup $\{Z(t): t \geq 0\}$ generated by A, then, is defined by

$$Z(t)f = \lim_{j \to \infty} Z_j(t)f \qquad (\forall f \varepsilon X, \ t \geq 0)$$

The final result in this section is a theorem concerning the inhomogeneous problem:

$$\frac{d}{dt} u(t) = Au(t) + g(t) \qquad (2.6)$$

$$u(0) = u^0 \qquad (2.7)$$

(2.4), (2.5)) is given by

$$u(t) = Z(t)u^0$$

for any $u^0 \varepsilon \mathcal{D}(A)$ and for all $t \geq 0$ $(-\infty < t < \infty)$.

Further Practical Results

Three results which are often of use in the application of the theory are now introduced. The first is generally useful if the underlying space X is a Hilbert space and the norm corresponds to the energy of the system.

Theorem 2.10

Let $A:\mathcal{D}(A) \rightarrow X$, where $\mathcal{D}(A) \subset X$ and $\mathcal{D}(A)$ is dense in the Hilbert space X. Then $A \varepsilon \mathfrak{G}(1,\beta)$ if and only if

(i) $(zI-A)\mathcal{D}(A) = X \quad \forall z$ such that $\mathrm{Re}(z) > \beta$

(ii) $\mathrm{Re}(Af,f) \leq \beta \|f\|^2 \quad \forall f \varepsilon \mathcal{D}(A)$

A densely defined linear operator A satisfying condition (ii) with $\beta = 0$ is called dissipative; also, if $-A$ is dissipative then A is called accretive. For further discussion in this area and a proof of Theorem 2.10, refer to (Belleni-Morante, 1979: 142-145).

Often a complicated operator A can be broken into two opeators: $A = A_b + A_u$. If A_b is chosen such that it is a bounded linear operator, defined on all of the underlying Banach space X, then the

$A \varepsilon \mathfrak{G}(M,\beta)$, $A \varepsilon \mathfrak{G}'(M,\beta)$. The proofs of Theorems 2.7 and 2.8 are easily modified for $A \varepsilon \mathfrak{G}(M,0)$ or $A \varepsilon \mathfrak{G}'(M,0)$. Furthermore, if $A \varepsilon \mathfrak{G}(M,\beta)$ $(A \varepsilon \mathfrak{G}'(M,\beta))$ then the operator A_1, defined by

$$A_1 = A - \beta I$$

is in the class $\mathfrak{G}(M,0)$ ($\mathfrak{G}'(M,0)$). Consequently, the following theorem can be proven with little additional work (Belleni-Morante, 1979: 159):

Theorem 2.9

If $A \varepsilon \mathfrak{G}(M,\beta)$, then A generates the strongly continuous semigroup $\{Z(t): t \geq 0\}$ with $Z(t)$ defined by

$$Z(t)u = e^{\beta t} S(t)u \qquad (\forall u \varepsilon X,\ t \geq 0)$$

where $\{S(t): t \geq 0\}$ is the semigroup generated by $A_1 = A - \beta I$.

The analogous result for $A \varepsilon \mathfrak{G}'(M,\beta)$ follows immediately. The norm of $Z(t)$ satisfies

$$\|Z(t)\| \leq Me^{\beta t} \qquad (-\infty < t < \infty)$$

for $A \varepsilon \mathfrak{G}(M,\beta)$, and, if $A \varepsilon \mathfrak{G}'(M,\beta)$ generates the strongly continuous group $\{Z(t): -\infty < t < \infty\}$,

$$\|Z(t)\| \leq Me^{\beta |t|} \qquad (-\infty < t < \infty)$$

In either case, the solution of equations (2.1), (2.2) (equations

Theorem 2.7

If $A \varepsilon \mathcal{G}(1,0)$, then A generates the strongly continuous semigroup $\{S(t) : t \geqq 0\}$ with $S(t)$ defined by

$$S(t)u = \lim_{n \to \infty} S_n(t)u \qquad (\forall u \varepsilon X, \ -\infty < t < \infty)$$

Additionally, this semigroup satisfies

$$\|S(t)\| \leqq 1$$

and, hence, the solution to the Cauchy problem of equations (2.1), (2.2) is again

$$u(t) = S(t)u^0 \qquad (t \geqq 0)$$

for any $u^0 \varepsilon \mathcal{D}(A)$.

Letting $S_n(t)$ be defined as above, but for $-\infty < t < \infty$, one also has the following (Belleni-Morante, 1979: 160):

Theorem 2.8

If $A \varepsilon \mathcal{G}'(1,0)$, then A generates the strongly continuous group $\{S(t) : -\infty < t < \infty\}$ with $S(t)$ defined by

$$S(t)u = \lim_{n \to \infty} S_n(t)u \qquad (\forall u \varepsilon X, \ -\infty < t < \infty)$$

The group thus defined satisfies

$$\|S(t)\| \leqq 1$$

and the solution of equations (2.4), (2.5) is

$$u(t) = S(t)u^0$$

for any $u^0 \varepsilon \mathcal{D}(A)$.

it is straightforward to show that f^* is a joint probability density function (pdf) for each $t \varepsilon [0,T]$ with random variables $\underline{x}$ and $\underline{p}$. Integration of f^* over all $\underline{p} \varepsilon R^3$ yields a marginal probability density function:

$$f^*_{\underline{x}}(\underline{x},t) = \int_{R^3} f^*(\underline{x},\underline{p},t)\, d^3p$$

This marginal pdf is related to the number density, $n(\underline{x},t)$, by

$$n(\underline{x},t) = N f^*_{\underline{x}}(\underline{x},t)$$

A conditional probability density function is now needed to express the macroscopic momentum vector, the macroscopic velocity vector, and the pressure tensor in terms of the probability space. Specifically, let the function $f^*_{\underline{p}|\underline{x}}$ be defined by

$$f^*_{\underline{p}|\underline{x}}(\underline{p};\underline{x},t) = \frac{f^*(\underline{x},\underline{p},t)}{f^*_{\underline{x}}(\underline{x},t)} = \frac{f(\underline{x},\underline{p},t)}{n(\underline{x},t)}$$

Since $f^*_{\underline{p}|\underline{x}}$ is a pdf for every $(x,t) \varepsilon R^3 \times [0,T]$, conditional expected values of any function $\theta(\underline{p})$ can be taken:

$$E[\theta(\underline{p})|\underline{x}] = \int_{R^3} \theta(\underline{p}) f^*_{\underline{p}|\underline{x}}(\underline{p};\underline{x},t)\ d^3p$$

Conditional expected values of the functions $\underline{p}$, $\underline{v}(\underline{p})$, and

$$[\underline{p} - \underline{P}(\underline{x},t)][\underline{v}(\underline{p}) - \underline{V}(\underline{x},t)]^T$$

yield $\underline{P}(\underline{x},t)$, $\underline{V}(\underline{x},t)$, and $\mathbb{P}(\underline{x},t)$, respectively.

Expression of the basic definitions of plasma physics in a probability theory setting provides rigor and clarity for applied mathematicians. On the other hand, the notation used by plasma physicists is both intuitive and well-established. Consequently, now that the connection between these two areas has been established, plasma physics notation and definitions are used in the remainder of this dissertation.

The rationalized MKS system of units is used in this chapter since this seems to be the choice of many authors of charged particle beam texts ((Lawson, 1977), for example). However, it should be noted that most plasma physics authors prefer the cgs Gaussian system (for example, see (Davidson, 1974; Krall and Trivelpiece, 1973)). Both systems have their advantges and disadvantages, and the transition from one system to the other is not difficult. In the rationalized MKS system, the symbols ε_0, μ_0 are used to represent the absolute dielectric constant and magnetic permeability which are related by

$$c^2 = \frac{1}{\mu_0 \varepsilon_0}$$

Finally, since vector cross products are somewhat tedious to write out in detail, the permutation symbol (Marion and Heald, 1980: 456), ε_{ijk}, as defined in Appendix B, and summation notation are frequently used. By way of example, consider the cross product

$$\underline{w} = \underline{u} \times \underline{v}$$

The components of $\underline{w}$ can be expresed compactly as

$$w_i = \varepsilon_{ijk} u_j v_k \qquad (i=1,2,3)$$

For instance, if $i=1$, the above expression yields

$$w_1 = \sum_{j=1}^{3} \sum_{k=1}^{3} \varepsilon_{1jk} = u_2 v_3 - u_3 v_2$$

since $\varepsilon_{123}=1$, $\varepsilon_{132}=-1$, and $\varepsilon_{1jk}=0$ for all other possible triples $(1,j,k)$.

Microscopic Descriptions

Vlasov Equation. The distribution function of a nonneutral collisionless plasma of a single species obeys the Vlasov equation (Davidson, 1974: 12). If $\underline{E}(\underline{x},t)$, $\underline{B}(\underline{x},t)$ represent the total electric and magnetic field at time t, the Vlasov equation in Cartesian coordinates can be written as

$$\frac{\partial}{\partial t} f(\underline{x},\underline{p},t) + v_i \frac{\partial}{\partial x_i} f(\underline{x},\underline{p},t)$$

$$+ q[E_i(\underline{x},t) + \varepsilon_{ijk} v_j B_k(\underline{x},t)] \frac{\partial}{\partial p_i} f(\underline{x},\underline{p},t) = 0 \qquad (3.6)$$

The fields $\underline{E}$, $\underline{B}$ arise from external charges as well as from collective effect from the particles in the plasma itself. Denoting the external fields by $\underline{E}^e$, $\underline{B}^e$ and the self fields by $\underline{E}^s$, $\underline{B}^s$, the total fields can be expressed as

$$\underline{E} = \underline{E}^e + \underline{E}^s$$

$$\underline{B} = \underline{B}^e + \underline{B}^s$$

Maxwell's Equations. Maxwell's equations must be satisfied as well as the Vlasov equation. The external fields are produced by external charges or current densities, but since these will ultimately be regarded as controls which can be applied in a prescribed manner, their corresponding Maxwell equations are unimportant at present. On the other hand, the self fields depend intimately upon the distribution function through Maxwell's equations:

$$\frac{\partial}{\partial t} E_i^s(\underline{x},t) = -\mu_0 c^2 \underline{J}(\underline{x},t) + c^2 \varepsilon_{ijk} \frac{\partial}{\partial x_j} B_k^s(\underline{x},t) \tag{3.7}$$

$$\frac{\partial}{\partial t} B_i^s(\underline{x},t) = -\varepsilon_{ijk} \frac{\partial}{\partial x_j} E_k^s(\underline{x},t) \tag{3.8}$$

$$\frac{\partial}{\partial x_j} E_j^s(\underline{x},t) = \sigma(\underline{x},t)/\varepsilon_0 \tag{3.9}$$

$$\frac{\partial}{\partial x_j} B_j^s(\underline{x},t) = 0 \tag{3.10}$$

for $i = 1, 2, 3$. Recalling that σ and $\underline{J}$ depend upon the distribution function f, it is seen that (3.6) through (3.10) represent a system of nine coupled nonlinear integro-differential equations. Representing the ordered set $(f(\underline{x},\underline{p},t), \underline{E}^s(\underline{x},t), \underline{B}^s(\underline{x},t))$ by $\underline{u}(t)$, equations (3.6) through (3.8) can be written as

$$\frac{d}{dt} \underline{u}(t) = \underline{F}(\underline{E}^e, \underline{B}^e)(\underline{u}(t)) \tag{3.11}$$

where $\underline{F}$ depends upon the external fields and represents the nonlinear operations indicated in those equations. Furthermore, equations (3.9), (3.10) serve as restrictions on the domain of $\underline{F}$, as would boundary conditions which are typically present in any given physical situation. A solution of the differential equation (3.11), and an associated initial value, $\underline{u}(0) = \underline{u}^0$, is generally difficult to obtain.

<u>Linearization of the System</u>. If the nonlinear operator is approximated by a linear operator, the resulting system can be shown to be an abstract Cauchy problem. This is now demonstrated for the special case of both the the electric and magnetic external fields being identically zero.

In infinite-dimensional systems, nonlinear operators can be approximated in a manner analogous to the first-order Taylor series technique in finite-dimensional systems. Consider the equation

$$\dot{\underline{x}}(t) = \underline{g}(\underline{x}(t))$$

where g is a vector-valued nonlinear function: $\underline{g}: R^n \to R^n$. If $\underline{x}^0$ is known to be a solution of $\underline{g}(\underline{x}^0) = 0$, and if

$$\frac{d}{dt}\underline{x}^0 = \underline{0}$$

then $\underline{x}^0$ is called an equilibrium solution. Suppose $\underline{g}$ can be represented by a Taylor series at $\underline{x}^0$:

$$\underline{g}(\underline{x}) = \underline{g}(\underline{x}^0) + \frac{\partial}{\partial \underline{x}} \underline{g}(\underline{x}^0)(\underline{x} - \underline{x}^0) + \ldots$$

Now, letting $\underline{\delta x}(t)=\underline{x}(t)-\underline{x}^0$, the original system can be approximated by the linearized perturbation equation

$$\dot{\underline{\delta x}}(t) \simeq J_{\underline{x}^0}\underline{\delta x}(t) \tag{3.12}$$

where $J_{\underline{x}^0} = \frac{\partial}{\partial \underline{x}} \underline{g}(\underline{x}^0)$. The operator $J_{\underline{x}^0}$ is the Frechet derivative of the nonlinear $\underline{g}$ at $\underline{x}^0$ provided each entry in $J_{\underline{x}^0}$ is continuous (see examples 1 and 4 of (Luenberger, 1969: 171-174)). In light of the comments in Chapter II following the discussion on Frechet derivatives, the Gateaux derivative of $\underline{g}$ is also $J_{\underline{x}^0}$.

In many situations it is not possible to show that an operator is Frechet differentiable, although the Gateaux derivative can usually be determined in a straightforward manner. Linearizations based on the Gateaux derivative cannot be justified rigorously a priori, but such models are often used. Solutions obtained for these models should be verified, if possible, by alternate methods.

Let the operator $\delta F_{u^0}:X\to X$, X a Banach space, represent the Gateaux derivative of $F:\mathcal{D}(F)\to X$ at u^0, and consider the equation

$$\frac{d}{dt}u(t) = F(u(t)) \tag{3.13}$$

Suppose u^0 is an equilibrium solution (defined in the same manner as in the finite-dimensional case: $F(u^0)=0$) and let $\delta u(t)=u(t)-u^0$. Provided the Gateaux derivative is a linear operator, the approximation

$$F(u^0+\delta u(t)) \approx F(u^0) + \delta F_{u^0}(\delta u(t)) \qquad (3.14)$$

is made. This is a direct result of the definition, since if $\delta u(t)=hv(t)$, for any $v\varepsilon X$, $h\varepsilon R$, the following limits exist:

$$\lim_{h\to 0} \frac{F(u^0+hv(t))-F(u^0)}{h} = \lim_{h\to 0} \frac{F(u^0+\delta u(t))-F(u^0)}{h} = \lim_{h\to 0} \frac{1}{h}\delta F_{u^0}\delta u(t)$$

By definition of the limit, then, this implies

$$\lim_{h\to 0} \frac{1}{h}\left\|F(u^0+\delta u(t))-F(u^0)-\delta F_{u^0}\delta u(t)\right\| = 0$$

Consequently, for h sufficiently small, the differential equation

$$\frac{d}{dt}\delta u(t) \approx \delta F_{u^0}\delta u(t)$$

becomes the infinite-dimensional analogue of the finite-dimensional result of equation (3.12).

With the approximation (3.14), the nonlinear Vlasov equation (3.6) can be linearized in a straightforward manner. Let $\underline{E}^e=\underline{B}^e=\underline{0}$, f^0, E^0 , B^0 be equilibrium solutions of equations (3.6) through (3.10), and define δf, $\underline{\delta E}$, $\underline{\delta B}$ in the obvious way. Then the linear approximation to the Vlasov equation is

$$\frac{\partial}{\partial t}\delta f(\underline{x},\underline{p},t) + v_i\frac{\partial}{\partial x_i}\delta f(\underline{x},\underline{p},t)$$

$$+ q(E_i^0(x) + \varepsilon_{ijk}v_jB_k^0(\underline{x},t))\frac{\partial}{\partial p_i}\delta f(\underline{x},\underline{p},t)$$

$$+ q(\delta E_i(\underline{x},\underline{p},t) + \varepsilon_{ijk}v_j\delta B_k(\underline{x},t)\frac{\partial}{\partial p_i}f^0(\underline{x},\underline{p}) = 0 \qquad (3.15)$$

Since Maxwell's equations are linear, simply replacing the functions (f, $\underline{E}^s$, $\underline{B}^s$) with (δf , $\underline{\delta E}^s$, $\underline{\delta B}^s$) in equations (3.7) through (3.10) yields the linearized versions of these equations:

$$\frac{\partial}{\partial t}\delta E_i^s(\underline{x},t) = -\mu_0 qc^2\int_{R^3} v_i\delta f(\underline{x},\underline{p},t)d^3p+c^2\varepsilon_{ijk}\frac{\partial}{\partial x_j}\delta B_k^s(\underline{x},t) \qquad (3.16)$$

$$\frac{\partial}{\partial t}\delta B_i^s(\underline{x},t) = -\varepsilon_{ijk}\frac{\partial}{\partial x_j}E_k^s(\underline{x},t) \qquad (3.17)$$

$$\frac{\partial}{\partial x_j}\delta E_j^s(\underline{x},t) = \frac{q}{\varepsilon_0}\int_{R^3}\delta f(\underline{x},\underline{p},t)d\ p \qquad (3.18)$$

$$\frac{\partial}{\partial x_j}\delta B_j^s(\underline{x},t) = 0 \qquad (3.19)$$

The system of equations (3.15), (3.16) and (3.17) represents an abstract Cauchy problem

$$\frac{d}{dt}\underline{w}(t) = A\underline{w}(t) \qquad (t>0)$$

with initial condition $\underline{w}(0) = \underline{w}^0$. The underlying Banach space $X = \prod_{i=1}^{7} X_i$ is yet to be specified. The domain of A should include the restrictions of equations (3.18), (3.19), as well as any additional boundary conditions.

Further analysis of the microscopic equations requires realistic equilibrium solutions which are smooth enough for their derivatives in equation (3.15) to exist. Such solutions are not known at this time, although equilibrium solutions involving the Dirac delta function have been discovered (e.g., see (Hammer and Rostoker, 1970: 1831-1834)). Various attempts were made to continue analysis of microscopic models using such equilibrium solutions, but the resulting linearized equations were intractable.

Macroscopic Descriptions

Equations can be developed from the Vlasov equation which describe the evolution of certain "averaged" quantities. Such equations are termed macroscopic descriptions, and the first two sets of these are presented below. These descriptions are appealing since the unknown functions associated with them are more intuitive than the distribution function in that the physical quantities involved are more directly observable. However, certain phenomena, such as Landau damping, cannot be predicted by such descriptions (Davidson, 1974: 11), and, consequently some information is forever lost once microscopic descriptions are abandoned.

The macroscopic equations are derived by multiplying the Vlasov equation by an appropriate function and integrating over all momentum space. Details are not presented here since they can be found in various plasma physics texts (see, for example, (Chen, 1974: 211-213)) The first two sets of equations are commonly referred to as the continuity equation (3.20) and the momentum equations (3.21) (Krall and Trivelpiece, 1973: 88; Davidson, 1974: 14):

$$\frac{\partial}{\partial t} n(\underline{x},t) = -\frac{\partial}{\partial x_i}[n(\underline{x},t)V_i(\underline{x},t)] \qquad (3.20)$$

$$\frac{\partial}{\partial t} P_i(\underline{x},t) + V_i(\underline{x},t)\frac{\partial}{\partial x_i} P_j(\underline{x},t) + \frac{1}{n(\underline{x},t)}\frac{\partial}{\partial x_i}[\mathbb{P}(\underline{x},t)]_{ij}$$

$$= q[E_j(\underline{x},t) + \varepsilon_{jkl}V_k(\underline{x},t)B_l(\underline{x},t)] \qquad (3.21)$$

where $j = 1,2,3$, and $[\mathbb{P}(\underline{x},t)]_{ij}$ is the (i,j)-component of the pressure tensor (see equation (3.5)).

Equations (3.20) and (3.21) cannot be solved without knowledge of the pressure tensor, $\mathbb{P}$. The components of $\mathbb{P}$ would appear as time derivatives in the next higher moment equation, the energy equation, but a quantity would be needed in this equation from the next higher moment equation, and so forth. This chain of moment equations is frequently broken here by making some approximation to $\mathbb{P}$. If the spread in the momentum is small at every point, then components of $\mathbb{P}$ are small and, in the limiting case of the momentum being a deterministic quantity everywhere, $\mathbb{P} = 0$ (Davidson, 1974: 16). The spread in velocity is also zero, in this case, and thus the temperature vanishes everywhere. This idealized case is termed the <u>cold plasma approximation</u>.

As mentioned previously, the macroscopic equations of (3.20) and (3.21) are somewhat more intuitive than the Vlasov equation which they replace. Integral operators are required in microscopic descriptions (see equations (3.7), (3.9)), but are unnecessary in the macroscopic descriptions. Furthermore, for the cold plasma approximation, although macroscopic descriptions replace a single unknown function (the distribution function, f) with four unknown functions (the number density and the three components of the macroscopic momentum vector) the reduction in the number of independent variables is three (from $(\underline{x},\underline{p},t)$ to $(\underline{x},t)$). For many applications, only these macroscopic functions are of interest. In light of these observations, it is concluded that macroscopic models are preferred in the design of particle beam control components, so long as they accurately describe the number density and the macroscopic momentum.

A Single Degree of Freedom Linear Model

Introduction. Various additional assumptions are introduced in this section in order to derive a suitable model for subsequent illustration of semigroup theory techniques. A broad variation in operating conditions exists for charged particle beams. Each assumption below has been invoked in plasma physics research in the investigation of beam behavior under a specific operating condition (e.g., see (Davidson, 1974)). Some assumptions, for example the nonrelativistic velocity assumption, serve only to call out a specific regime of operation. Other assumptions, such as the assumption of the adequacy of macroscopic descriptions, are made to simplify the model, with the justification being that they have previously been invoked by plasma physics researchers and have been found to be useful and adequate in describing beam dynamic phenomena. In either case, the philosophy taken now is that simple, though less accurate models whose dominant behavior can be expressed analytically, are superior in preliminary control designs to more precise models which require computer generated numerical solutions.*

The cold plasma assumption is in keeping with this philosophy. As previously stated, this is equivalent to assuming that the momentum is deterministic at every point. In practice, if the momentum spread at every point is sufficiently small, then the cold plasma assumption is reasonable. Otherwise, approximations of the pressure tensor might be required (see (Krall and Trivelpiece, 1973: 89)).

* The assumptions made in no way limit the applicability of semigroup theory. For example, the same techniques used below to analyze the single degree-of-freedom model could be applied to a three degree-of-freedom model. See Appendix D for the structure of such a model.

Reduction in the number of degrees of freedom is also useful in simplifying the model. However, models which allow only a single spatial degree of freedom in a Cartesian coordinate system are generally unrealistic, so a cylindrical coordinate system is introduced.

Since most applications of a charged particle beam require only that the beam operate near some design equilibrium solution, a linear model about an equilibrium point should be adequate for control purposes. Indeed, if the control function desired is that of regulation, deviations from the equilibrium will be constrained to be small by the action of the regulator. This notion is fundamental to control theory design and has been applied with success routinely over the years.

Assumptions. The following assumptions are in effect in the development below:

A1. the beam is in a vacuum

A2. all velocities are nonrelativistic

A3. the momentum spread at each point is small (cold plasma)

A4. macroscopic descriptions are adequate

A5. the beam is uniform in the azimuthal direction

A6. the beam is uniform in the axial direction

A7. deviations from the equilibrium solution are small

Additional assumptions are needed later (page III-24) for the development of a specific equilibrium solution, and are stated at the outset of that development.

Assumptions (A1) through (A4) result in the following system of equations (cf., equations (3.20) and (3.21)):

$$\frac{\partial}{\partial t} n(\underline{x},t) = -\frac{\partial}{\partial x_i}[n(\underline{x},t)V_i(\underline{x},t)] \tag{3.22}$$

$$\frac{\partial}{\partial t} V_j(\underline{x},t) = -V_i(\underline{x},t)\frac{\partial}{\partial x_i}V_j(\underline{x},t)+\frac{q}{m}[E_j(\underline{x},t)+\epsilon_{ijk}V_k(\underline{x},t)B_l(\underline{x},t)] \tag{3.23}$$

for $j = 1,2,3$ and (cf., equations (3.7) through (3.10)),

$$\frac{\partial}{\partial t} E_i^S(\underline{x},t) = -4\pi q n(\underline{x},t)V_i(\underline{x},t)+c\epsilon_{ijk}\frac{\partial}{\partial x_j}B_k^S(\underline{x},t) \tag{3.24}$$

$$\frac{\partial}{\partial t} B_i^S(\underline{x},t) = -c\epsilon_{ijk}\frac{\partial}{\partial x_j}E_k^S(\underline{x},t) \tag{3.25}$$

$$\frac{\partial}{\partial x_i} E_i^S(\underline{x},t) = 4\pi q n(\underline{x},t) \tag{3.26}$$

$$\frac{\partial}{\partial x_i} B_i^S(\underline{x},t) = 0 \tag{3.27}$$

[illegible] (3.[illegible]) and (3.[illegible]) [illegible] in a cylindrical [illegible]

[illegible], the [illegible] of [illegible] at $\underline{z}^0 \varepsilon \mathcal{Y}\times\mathcal{X}$, $\underline{\delta G}_{\underline{z}^0}$, can be computed as follows:

$$\underline{\delta G}_{\underline{z}^0}(\underline{\zeta}) = \lim_{h\to 0} \frac{\underline{G}(\underline{z}^0+h\underline{\zeta}) - \underline{G}(\underline{z}^0)}{h}$$

$$= \frac{q}{m}\begin{bmatrix} 0 \\ \zeta_1 + z_9^0\zeta_6 + z_6^0\zeta_9 - (z_{10}^0\zeta_5 + z_5^0\zeta_{10}) \\ \zeta_2 + z_{10}^0\zeta_4 + z_4^0\zeta_{10} - (z_8^0\zeta_6 + z_6^0\zeta_8) \\ \zeta_3 + z_8^0\zeta_5 + z_5^0\zeta_8 - (z_9^0\zeta_4 + z_4^0\zeta_9) \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad (3.66)$$

This expression represents the Gateaux derivative of $\underline{G}$ for a general equilibrium $\underline{z}^{\circ}$ and associated external equilibrium fields, $\underline{E}^{e,0}$, $\underline{B}^{e,0}$. For the rigid rotor equilibrium, the vector $\underline{z}^0$ is given by

$$\underline{G}(\underline{E}^e, \underline{B}^e)(\underline{\mu}) = \frac{q}{m} \begin{bmatrix} 0 \\ E_r^e(r,t) + B_z^e(r,t)\mu_3 - B_\vartheta^e(r,t)\mu_4 \\ E_\vartheta^e(r,t) + B_r^e(r,t)\mu_4 - B_z^e(r,t)\mu_2 \\ E_z^e(r,t) + B_\vartheta^e(r,t)\mu_2 - B_r^e(r,t)\mu_3 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad (3.65)$$

Since the external fields $\underline{E}^e$, $\underline{B}^e$ are being viewed as the controls, define the function space consisting of all physically attainable controls as $\mathcal{Y} = \prod_{i=1}^{6} Y_i$ where $E_r^e \varepsilon Y_1$, $E_\vartheta^e \varepsilon Y_2$, $\ldots B_z^e \varepsilon Y_6$. Consequently, the mapping $\underline{G}(\underline{E}^e, \underline{B}^e)$ represents a unique nonlinear operator for every $(E^e, \underline{B}^e)\varepsilon\mathcal{Y}$ or, alternatively, one can consider the mapping $\underline{G}$ as one which takes elements from $\mathcal{Y}\times\mathcal{X}$ into $\mathcal{X}$. This latter interpretation allows an approximation of $\underline{G}$ by the Gateaux derivative in a manner analogous to the approximation above for $\underline{F}$.

Consider an arbitrary element $\underline{\zeta}$ in $\mathcal{Y}\times\mathcal{X}$, and suppose the X_i, $i = 1,2\ldots9$, are all Banach spaces. The Gateaux derivative of $\underline{G}$

$$\underline{\delta F}_{u^0}(\underline{\eta}) =$$

$$\begin{bmatrix} -n^0 \tilde{D}_r \eta_2 \\ -2\omega\eta_3 - \frac{\omega_p^2 V_z^0}{2c^2} r\eta_4 + \frac{q}{m}\eta_5 - \frac{qV_z^0}{m}\eta_8 - \frac{q\omega}{m} r\eta_9 \\ 2\omega\eta_2 + \frac{q}{m}\eta_6 \\ \frac{\omega_p^2 V_z^0}{2c^2} r\eta_2 + \frac{q}{m}\eta_7 \\ -\mu_0 q c^2 n^0 \eta_2 \\ \mu_0 q c^2 \omega r \eta_1 - \mu_0 q c^2 n^0 \eta_3 - c^2 D_r \eta_9 \\ -\mu_0 q c^2 V_z^0 \eta_1 - \mu_0 q c^2 n^0 \eta_4 + c^2 \tilde{D}_r \eta_8 \\ D_r \eta_7 \\ -\tilde{D}_r \eta_6 \end{bmatrix} \quad (3.64)$$

where D_r and $\tilde{D}_r$ are defined by

$$D_r f = \frac{d}{dr} f$$

$$\tilde{D}_r f = \frac{1}{r}\frac{d}{dr}(rf)$$

Consider next the nonlinear operator $\underline{G}(\underline{E}^e, \underline{B}^e)$:

$$\underline{\delta F}_{\underline{u}^0}(\underline{\eta}) = \lim_{h\to 0} \frac{\underline{F}(\underline{u}^0+h\underline{\eta}) - \underline{F}(\underline{u}^0)}{h}$$

$$= \begin{bmatrix} -\left(\frac{1}{r}\frac{d}{dr}(ru_2^0) + u_2^0\frac{d}{dr}\right)\eta_1 - \left(\frac{1}{r}\frac{d}{dr}(ru_1^0) + u_1^0\frac{d}{dr}\right)\eta_2 \\ -\left(\frac{d}{dr}u_2^0 + u_2^0\frac{d}{dr}\right) + \left(\frac{2}{r}u_3^0 + \frac{q}{m}u_9^0\right)\eta_3 - \frac{q}{m}u_8^0\eta_4 \\ \qquad\qquad + \frac{q}{m}\eta_5 - \frac{q}{m}u_4^0\eta_8 + \frac{q}{m}u_3^0\eta_9 \\ -\left(\frac{1}{r}\frac{d}{dr}ru_3^0 + \frac{q}{m}u_9^0\right)\eta_2 - \left(\frac{1}{r}u_2^0 + u_2^0\frac{d}{dr}\right)\eta_3 + \frac{q}{m}\eta_6 - \frac{q}{m}u_2^0\eta_9 \\ \left(\frac{q}{m}u_8^0 - \frac{d}{dr}u_4^0\right)\eta_2 - u_2^0\frac{d}{dr}\eta_4 + \frac{q}{m}\eta_7 + \frac{q}{m}u_2^0\eta_8 \\ -\mu_0 qc^2(u_2^0\eta_1 + u_1^0\eta_2) \\ -\mu_0 qc^2(u_3^0\eta_1 + u_1^0\eta_3) - c^2\frac{d}{dr}\eta_9 \\ -\mu_0 qc^2(u_4^0\eta_1 + u_1^0\eta_4) + \frac{c^2}{r}\frac{d}{dr}r\eta_8 \\ \frac{d}{dr}\eta_7 \\ -\frac{1}{r}\frac{d}{dr}r\eta_6 \end{bmatrix} \tag{3.63}$$

This expression is the Gateaux derivative for $\underline{F}$ for a general equilibrium solution $\underline{u}^0$. For the rigid rotor equilibrium (equation (3.60)), ments of $\underline{u}(t)$ in equations (3.41),(3.42), and (3.43) which involve the external fields, and it is discussed in more detail presently. A linear model which approximates this system is now developed.

Let $\underline{u}^0$, B^0 continue to denote the rigid rotor equilibrium solution (equation (3.60)), and the z-component of the external, uniform magnetic field, respectively. Define the perturbed variables $\underline{\delta u}(t)$ in the usual way: $\underline{\delta u}(t) = \underline{u}(t) - \underline{u}^0$. Similarly, let the perturbed external fields, $\underline{\delta E}^e(r,t)$, $\underline{\delta B}^e(r,t)$ be given by

$$\underline{\delta E}^e(r,t) = \underline{E}^e(r,t) - \underline{E}^{e,0}(r) = \underline{E}^e(r,t)$$

$$\underline{\delta B}^e(r,t) = \underline{B}^e(r,t) - \underline{B}^{e,0}(r) = \begin{bmatrix} B_r^e(r,t) \\ B_\theta^e(r,t) \\ B_z^e(r,t) - B^0 \end{bmatrix}$$

If all the spaces X_i, $i = 1,2,\ldots 9$, are Banach spaces, then the nonlinear operator $\underline{F}$ can be approximated by

$$\underline{F}(\underline{u}^0 + \underline{\delta u}(t)) \simeq \underline{F}(\underline{u}^0) + \underline{\delta F}_{\underline{u}^0}(\underline{\delta u}(t)) \tag{3.62}$$

where $\underline{\delta F}_{\underline{u}^0}$ is the (linear) Gateaux derivative of $\underline{F}$ at $\underline{u}^0$ (see the discussion associated with equation (3.14) in the section "Microscopic Descriptions"). Calculation of $\underline{\delta F}_{\underline{u}^0}$ is straightforward:

1973: 117). Finally, for $\omega_c^2 > 2\omega_p^2$, two equilibria are possible (termed the slow mode/ fast mode (Davidson, 1974: 7)).

The complete solution for $r\varepsilon(0,R]$ is now summarized. Letting $\underline{u}^0 = [n^0(r), V_r^0(r), V_\vartheta^0(r), V_z^0(r), E_r^0(r), E_\vartheta^0(r), E_z^0(r), B_\vartheta^0(r), B_z^0(r)]^T$, the rigid rotor equilibrium solution is

$$\underline{u}^0 = \begin{bmatrix} n^0 \\ 0 \\ -\omega r \\ V_z^0 \\ \frac{qn^0}{2\varepsilon_0}r \\ 0 \\ 0 \\ \frac{\mu_0 q n^0 V_z^0}{2}r \\ 0 \end{bmatrix} \tag{3.60}$$

<u>Linearization</u>. Equations (3.40) through (3.46) and (3.48) through (3.50) are seen to form a system of nine coupled nonlinear differential equations and can be expressed as

$$\frac{d}{dt}\underline{u}(t) = \underline{F}(\underline{u}(t)) + \underline{G}(\underline{E}^e, \underline{B}^e)(\underline{u}(t)) \tag{3.61}$$

where $\underline{F}:\mathcal{D}(\underline{F})\subset\mathfrak{X}\to\mathfrak{X}$, $\mathfrak{X} = \prod_{i=1}^{9} X_i$, and $\underline{u}(t) = [n(r,t), V_r(r,t), V_\vartheta(r,t), V_z(r,t), E_r(r,t), E_\vartheta(r,t), E_z(r,t), B_\vartheta(r,t), B_z(r,t)]^T$. The operator $\underline{G}(\underline{E}^e, \underline{B}^e)$ represents operations on ele-

The term qB^0/m, commonly called the cyclotron frequency (Lawson, 1977: 17), appears often in plasma physics and is denoted by ω_c. The term within the brackets is simplified by recalling that

$$c^2 = \frac{1}{\mu_0 \varepsilon_0}$$

and thus, this term reduces to

$$\frac{q^2 n^0}{2m\varepsilon_0}[1 - (\beta_z^0)^2]$$

By the nonrelativistic assumption, $(\beta_z^0)^2$ is neglected. The phrase plasma frequency is given to the expression (Lawson, 1977: 119)

$$|q|\,[\frac{n^0}{m\varepsilon_0}]^{\frac{1}{2}}$$

and it is given the symbol ω_p. The following expression for $V_\theta^0(r)$ emerges in light of the foregoing:

$$V_\theta^0(r) = -\frac{1}{2}[\omega_c \pm [\omega_c^2 - 2\omega_p^2]^{\frac{1}{2}}] \tag{3.59}$$

The use of the phrase "rigid rotor equilibrium" is justified by this expression.

The expression within the radical in equation (3.59) provides a minimum value for B^0. The external magnetic field is "sufficiently large" (see assumption E1 above) if

$$\omega_c^2 \geqq 2\omega_p^2 \quad > B^0 \geqq [\frac{2mn^0}{\varepsilon_0}]^{\frac{1}{2}}$$

If $\omega_c = 2\omega_p^2$, then the rotation rate is $\frac{\omega_c}{2}$, and the phrase "Brillouin flow" is used to describe this situation (Krall and Trivelpiece,

$$\frac{d}{dr} r B_\vartheta^{s,0}(r) = \begin{cases} \mu_0 q (V_z^0)^2 r & 0 \leq r \leq R \\ 0 & r > R \end{cases}$$

$$\frac{d}{dr} r E_r^{s,0}(r) = \begin{cases} \dfrac{qn^0}{\varepsilon_0} r & 0 \leq r \leq R \\ 0 & r > R \end{cases}$$

The solutions of these equations are

$$B_\vartheta^{s,0}(r) = \begin{cases} \dfrac{\mu_0 q n^0 V_z^0}{2} r & 0 \leq r \leq R \\ \dfrac{\mu_0 q n^0 V_z^0 R}{2} \dfrac{R}{r} & r > R \end{cases}$$

$$E_r^{s,0}(r) = \begin{cases} \dfrac{qn^0}{2\varepsilon_0} r & 0 \quad r \quad R \\ \dfrac{qn^0 R}{2\varepsilon_0} \dfrac{R}{r} & r > R \end{cases}$$

Applying these solutions to equation (3.52) for $r \varepsilon (0, R]$ one obtains

$$V_\vartheta^0(r)^2 + \frac{q^2 n^0}{2m\varepsilon_0} r^2 + \frac{qB^0}{m} r V_\vartheta^0(r) - \frac{\mu_0 q (V_z^0)^2 n^0}{2m} r^2 = 0$$

Solving this quadratic equation for $V_\vartheta^0(r)$ yields

$$V_\vartheta^0(r) = -\frac{qB^0}{2m} r \pm \frac{1}{2}\left\{\left[\frac{qB^0}{m}\right]^2 r^2 - 4\left[\frac{q^2 n^0}{2m\varepsilon_0} - \frac{\mu_0 q^2 (V_z^0)^2 n^0}{2m}\right] r^2\right\}^{1/2} \quad (3.58)$$

must satisfy).

The final assumption simplifies the solution considerably and is realistic so long as the beam is nonrelativistic (Krall and Trivelpiece, 1973: 117).

Assumptions (E1) through (E5) are now applied to equations (3.40) through (3.51). First, note that equations (3.40), (3.42), (3.43), (3.44), (3.47), and (3.51) are all trivially satisfied. Letting the superscript "0" denote an equilibrium solution function, the remaining equations become

$$\frac{V_\theta^0(r)^2}{r} + \frac{q}{m}[E_r^{s,0}(r)+V_\theta^0(r)B_z^{e,0}(r)-V_z^0(r)B_\theta^{s,0}(r)] = 0 \qquad (3.52)$$

$$E_\theta^{s,0}(r) = 0 \qquad (3.53)$$

$$E_z^{s,0}(r) = 0 \qquad (3.54)$$

$$\mu_0 qc^2V_z^0(r) + c^2\frac{d}{dr}B_z^{s,0}(r) = 0 \qquad (3.55)$$

$$-\mu_0 qc^2V_z^0(r) + \frac{c^2}{r}\frac{d}{dr}(rB_\theta^{s,0}(r)) = 0 \qquad (3.56)$$

$$\frac{1}{r}\frac{d}{dr}(rE_r^{s,0}(r)) = \frac{q}{\varepsilon_0}n^0(r) \qquad (3.57)$$

Denote the constant external magnetic field, number density, and z component of velocity by B^0, n^0, and V_z^0, respectively. The last two equations become

Nonrelativistic Rigid-Rotor Equilibrium. Consider a beam in the shape of a long cylinder of circular cross-section with radius R, the axis of which coincides with the z axis of a cylindrical coordinate system. Certain assumptions are required, in addition to those previously stated, for the rigid rotor equilibrium solution:

E1. a (sufficiently large) uniform magnetic field in the z direction is the only external field

E2. the number density is constant for $0 \leq r \leq R$, and vanishes on $r > R$

E3. the velocity in the z direction is constant for $0 \leq r \leq R$

E4. the radial velocity is identically zero

E5. the z component of the self magnetic field is negligible compared to the external field

Some discussion of these additional assumptions is now in order.

The external field in assumption (E1) is necessary to offset the repulsive forces which would cause the beam to expand indefinitely. The particles undergo a helical motion in the presence of this magnetic field, resulting in a balance between the repulsive forces (electrostatic and centrifugal), and the constrictive force (magnetic) whenever the external magnetic field is sufficiently large. Other means of confining a beam to a finite radius are possible (such as by neutralization by background ions (Lawson, 1977: 258), for example).

Assumptions (E2), (E3), and (E4) represent a simple configuration of the beam which may be useful in applications. Other combinations of number density, axial and azimuthal velocities are possible, however (see (Davidson, 1974: 20, 21) for a general equation which these

$$\frac{\partial}{\partial t} B_r^s = 0 \tag{3.47}$$

$$\frac{\partial}{\partial t} B_\vartheta^s = \frac{\partial}{\partial r} E_z^s \tag{3.48}$$

$$\frac{\partial}{\partial t} B_z^s = -\frac{1}{r} \frac{\partial}{\partial r} r E_\vartheta^s \tag{3.49}$$

$$\frac{1}{r} \frac{\partial}{\partial r} r E_r^s = \frac{q}{\varepsilon_0} n \tag{3.50}$$

$$\frac{1}{r} \frac{\partial}{\partial r} r B_r^s = 0 \tag{3.51}$$

Note that equation (3.47) implies B_r^s depends only upon the value of r , and that equation (3.51) implies that the form of B_r^s is

$$B_r^s(r,t) = \frac{K}{r} \qquad (r>0)$$

where K is an arbitrary constant. If $B_r^s(r,t)$ is to remain bounded as $r \to 0$, then $K=0$ and, thus,

$$B_r^s(r,t) = 0 \qquad (r>0, \ t \geq 0)$$

Thus far assumptions (A1) through (A6) have been implemented. Assumption (A7) requires development of a specific equilibrium solution. The "rigid-rotor" equilibrium (Davidson, 1974: 30; Krall and Trivelpiece, 1973: 116, 117) is well known in plasma physics. A derivation of this equilibrium solution for a nonrelativistic plasma is now given. Use of this specific equilibrium is not required in general, however, since any suitable equilibrium, analytically or numerically derived, is suitable for the linearization process.

apply assumptions (A5) and (A6). For these reasons vector notation is not used here.

Assumptions (A5) and (A6), uniformity in the θ and z directions, respectively, are now invoked by neglecting terms in equations (3.28) through (3.39) which involve $\frac{\partial}{\partial\theta}$ and $\frac{\partial}{\partial z}$. This results in the following nonlinear system of equations (the argument (r,t) has been dropped for notational convenience):

$$\frac{\partial}{\partial t} n = -\frac{1}{r}\frac{\partial}{\partial r}(nV_r) \tag{3.40}$$

$$\frac{\partial}{\partial t} V_r = -V_r\frac{\partial}{\partial r}V_r + \frac{V_\theta^2}{r} + \frac{q}{m}[E_r^s + V_\theta B_z^s - V_z B_\theta^s] + \frac{q}{m}[E_r^e + V_\theta B_z^e - V_z B_\theta^e] \tag{3.41}$$

$$\frac{\partial}{\partial t} V_\theta = -V_r\frac{\partial}{\partial r}V_\theta - \frac{V_r V_\theta}{r} + \frac{q}{m}[E_\theta^s + V_z B_r^s - V_r B_z^s] + \frac{q}{m}[E_\theta^e + V_z B_r^e - V_r B_z^e] \tag{3.42}$$

$$\frac{\partial}{\partial t} V_z = -V_r\frac{\partial}{\partial r}V_z + \frac{q}{m}[E_z^s + V_r B_\theta^s - V_\theta B_r^s] + \frac{q}{m}[E_z^e + V_r B_\theta^e - V_\theta B_r^e] \tag{3.43}$$

$$\frac{\partial}{\partial t} E_r^s = -\mu_0 q c^2 n V_r \tag{3.44}$$

$$\frac{\partial}{\partial t} E_\theta^s = -\mu_0 q c^2 n V_\theta - c^2\frac{\partial}{\partial r}B_z^s \tag{3.45}$$

$$\frac{\partial}{\partial t} E_z^s = -\mu_0 q c^2 n V_z + \frac{c^2}{r}\frac{\partial}{\partial r}(rB_\theta^s) \tag{3.46}$$

$$\frac{\partial}{\partial t}V_z = -V_r\frac{\partial}{\partial r}V_z - \frac{V_\vartheta}{r}\frac{\partial}{\partial\vartheta}V_z - V_z\frac{\partial}{\partial z}V_z$$

$$+ \frac{q}{m}[E_z^s + V_r B_\vartheta^s - V_\vartheta B_r^s] + \frac{q}{m}[E_z^e + V_r B_\vartheta^e - V_\vartheta B_r^e] \tag{3.31}$$

$$\frac{\partial}{\partial t}E_r^s = -\mu_0 q c^2 V_r + c^2[\frac{1}{r}\frac{\partial}{\partial\vartheta}B_z^s - \frac{\partial}{\partial z}B_\vartheta^s] \tag{3.32}$$

$$\frac{\partial}{\partial t}E_\vartheta^s = -\mu_0 q c^2 V_\vartheta + c^2[\frac{\partial}{\partial z}B_r^s - \frac{\partial}{\partial r}B_z^s] \tag{3.33}$$

$$\frac{\partial}{\partial t}E_z^s = -\mu_0 q c^2 V_z + c^2[\frac{1}{r}\frac{\partial}{\partial r}rB_\vartheta^s - \frac{1}{r}\frac{\partial}{\partial\vartheta}B_r^s] \tag{3.34}$$

$$\frac{\partial}{\partial t}B_r^s = \frac{\partial}{\partial z}E_\vartheta^s - \frac{1}{r}\frac{\partial}{\partial\vartheta}E_z^s \tag{3.35}$$

$$\frac{\partial}{\partial t}B_\theta^s = \frac{\partial}{\partial r}E_z^s - \frac{\partial}{\partial z}E_r^s \tag{3.36}$$

$$\frac{\partial}{\partial t}B_z^s = \frac{1}{r}\frac{\partial}{\partial\vartheta}E_r^s - \frac{1}{r}\frac{\partial}{\partial r}rE_\vartheta^s \tag{3.37}$$

$$\frac{1}{r}\frac{\partial}{\partial r}rE_r^s + \frac{1}{r}\frac{\partial}{\partial\vartheta}E_\vartheta^s + \frac{\partial}{\partial z}E_z^s = \frac{q}{\varepsilon_\vartheta}n \tag{3.38}$$

$$\frac{1}{r}\frac{\partial}{\partial r}rB_r^s + \frac{1}{r}\frac{\partial}{\partial\vartheta}B_\vartheta^s + \frac{\partial}{\partial z}B_z^s = 0 \tag{3.39}$$

A far more compact statement of these equations is usually given in plasma physics texts by the use of vector notation (see (Krall and Trivelpiece, 1973: 85, 86; Davidson, 1974: 14, 15; Shkarofsky et al, 1963: 12, 21; Montgomery and Tidman, 1964: 12)). These compact forms, however, can be confusing to those not familiar with the notation. Furthermore, the detailed expressions above are needed in order to orthogonal system which has been rotated about the z-axis by the angle ϑ:

$$T_x^r = \begin{bmatrix} \cos\vartheta & \sin\vartheta & 0 \\ -\sin\vartheta & \cos\vartheta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Letting $[\psi_1, \psi_2, \psi_3]^T$ represent $[V_1, V_2, V_3]^T$, $[E_1, E_2, E_3]^T$, or $[B_1, B_2, B_3]^T$, then, the new functions $[\psi_r, \psi_\vartheta, \psi_z]^T$ are defined by

$$\begin{bmatrix} \psi_r \\ \psi_\vartheta \\ \psi_z \end{bmatrix} = T_x^r \begin{bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{bmatrix}$$

Using these definitions, equations (3.22) through (3.27) can be expressed in terms of the new unknown functions n, V_r, V_ϑ, V_z, E_r^s, E_ϑ^s, E_z^s, B_ϑ^s, B_z^s, with independent variables (r, ϑ, z, t) (the argument $(\underline{r}, t)$ is dropped for notational convenience):

$$\frac{\partial}{\partial t} n = -\frac{1}{r}\left(nV_r\right) - \frac{\partial}{\partial r} nV_r - \frac{1}{r}\frac{\partial}{\partial\vartheta} nV_\vartheta - \frac{\partial}{\partial z} nV_z \tag{3.28}$$

$$\begin{aligned} \frac{\partial}{\partial t} V_r = & -V_r \frac{\partial}{\partial r} V_r - \frac{V_\vartheta}{r}\frac{\partial}{\partial\vartheta} V_r + \frac{V_\vartheta^2}{r} - V_z \frac{\partial}{\partial z} V_r \\ & + \frac{q}{m}[E_r^s + V_\vartheta B_z^s - V_z B_\vartheta^s] + \frac{q}{m}[E_r^e + V_\vartheta B_z^s - V_z B_\vartheta^s] \end{aligned} \tag{3.29}$$

$$\begin{aligned} \frac{\partial}{\partial t} V_\vartheta = & -V_r \frac{\partial}{\partial r} V_\vartheta - \frac{V_\vartheta}{r}\frac{\partial}{\partial\vartheta} V_\vartheta - \frac{V_r V_\vartheta}{r} - V_z \frac{\partial}{\partial z} V_z \\ & + \frac{q}{m}[E_\vartheta^s + V_z B_r^s - V_r B_z^s] + \frac{q}{m}[E_\vartheta^e + V_z B_r^e - V_r B_z^e] \end{aligned} \tag{3.30}$$

$$
\underset{\sim}{z}^0 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ B^0 \\ \cdots\cdots \\ \underline{u}^0 \end{bmatrix}
$$

with $\underline{u}^0$ explicitly defined in equation (3.60). The operator reduces considerably, in this case, as follows:

$$
\underline{\delta G}_{\underline{z}^0}(\underline{\zeta}) = \frac{q}{m} \begin{bmatrix} 0 \\ \zeta_1 - \omega r \zeta_6 - V_z^0 \zeta_5 \\ \zeta_2 + V_z^0 \zeta_4 \\ \zeta_3 + \omega r \zeta_4 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \frac{q}{m} \begin{bmatrix} 0 \\ B^0 \zeta_9 \\ -B^0 \zeta_8 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad (3.67)
$$

The nonlinear $\underline{G}$ can be approximated, then, by

$$\underline{G}(\underline{E}^{e},\underline{B}^{e},\underline{u}(t)) \simeq \underline{G}(\underline{E}^{e,0},\underline{B}^{e,0},\underline{u}^{0}) + \underline{\delta G}_{\underline{z}^0}(\underline{\delta E}^{e},\underline{\delta B}^{e},\underline{\delta u}(t)) \quad (3.68)$$

Using the approximations of equations (3.62), (3.68) in equation (3.61) yields

$$\frac{d}{dt}\,\underline{u}(t) = \frac{d}{dt}(\underline{u}^{0} + \underline{\delta u}(t)) \simeq \underline{F}(\underline{u}^{0}) + \underline{G}(\underline{E}^{e,0},\underline{B}^{e,0},\underline{u}^{0})$$

$$+ \underline{\delta F}_{\underline{u}^0}(\underline{\delta u}(t)) + \underline{\delta G}_{\underline{z}^0}(\underline{\delta E}^{e},\underline{\delta B}^{e},\underline{\delta u}(t))$$

This implies, then, that

$$\frac{d}{dt}\underline{\delta u}(t) \simeq \underline{\delta F}_{\underline{u}^0}(\underline{\delta u}(t)) + \underline{\delta G}_{\underline{z}^0}(\underline{\delta E}^{e},\underline{\delta B}^{e},\underline{\delta u}(t)) \quad (3.69)$$

This equation represents a linear approximation to the nonlinear system of equations (3.40) through (3.46) and (3.48) through (3.50). A similar model can be developed for the region $r>R$. In general, these two models must be solved simultaneously, and their solutions must satisfy further mathematical constraints at $r=R$. Consequently, only small excursions of the actual beam radius from the equilibrium solution radius are allowed before a relinearization must be performed.

Linear Model. To summarize the development thus far, the expressions derived in equations (3.64) and (3.67), based on the rigid rotor assumptions, are substituted into equation (3.69) to yield

$$\frac{d}{dt}\underline{\delta u}(t) \simeq \begin{bmatrix} -n^0\tilde{D}_r\delta u_2(t) \\ \left(-2\omega\delta u_3(t) - \frac{\omega_p^2 V_z^0}{2c^2} r\delta u_4(t) + \frac{q}{m}\delta u_5(t) \right. \\ \left. - \frac{qV_z^0}{m}\delta u_8(t) - \frac{q\omega}{m} r\delta u_9(t)\right) \\ 2\omega\delta u_2(t) + \frac{q}{m}\delta u_6(t) \\ \frac{\omega_p^2 V_z^0}{2c^2} r\delta u_2(t) + \frac{q}{m}\delta u_7(t) \\ -\mu_0 qc^2 n^0\delta u_2(t) \\ \mu_0 qc^2\omega r\delta u_1(t) - \mu_0 qc^2 n^0\delta u_3(t) - c^2 D_r\delta u_9(t) \\ -\mu_0 qc^2 V_z^0\delta u_1(t) - \mu_0 qc^2 n^0\delta u_4(t) + c^2\tilde{D}_r\delta u_8(t) \\ D_r\delta u_7(t) \\ -\tilde{D}_r\delta u_6(t) \end{bmatrix}$$

$$+ \frac{q}{m}\begin{bmatrix} 0 \\ B^0\delta u_3(t) \\ -B^0\delta u_2(t) \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \frac{q}{m}\begin{bmatrix} 0 \\ \delta E_r^e(r,t) - \omega r\delta B_z^e(r,t) - V_z^0\delta B_\theta^e(t) \\ \delta E_\theta^e(r,t) + V_z^0\delta B_r^e(r,t) \\ \delta E_z^e(r,t) + \omega r\delta B_r^e(r,t) \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad (3.70)$$

This equation suggests a linear model of the form

$$\frac{d}{dt}\underline{w}(t) = A\underline{w}(t) + \underline{g}(t) \tag{3.71}$$

where

$$A =$$

$$\begin{bmatrix}
0 & -n^0\tilde{D}_r & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \omega_c - 2\omega & -\frac{\omega_p^2 V^0}{2c^2}z_r & \frac{q}{m} & 0 & 0 & -\frac{qV^0}{m}z & -\frac{q\Omega}{m}r \\
0 & 2\omega - \omega_c & 0 & 0 & 0 & \frac{q}{m} & 0 & 0 & 0 \\
0 & \frac{\omega_p^2 V^0}{2c^2}z_r & 0 & 0 & 0 & 0 & \frac{q}{m} & 0 & 0 \\
0 & \text{[illegible]} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\text{[illegible]} & 0 & \text{[illegible]} & 0 & 0 & 0 & 0 & 0 & -c^2 D_r \\
\text{[illegible]} & 0 & 0 & \text{[illegible]} & 0 & 0 & 0 & c^2\tilde{D}_r & 0 \\
0 & 0 & \text{[illegible]} & 0 & 0 & 0 & \text{[illegible]} & 0 & 0 \\
\text{[illegible]} & 0 & \text{[illegible]} & \text{[illegible]} & \text{[illegible]} & \text{[illegible]} & \text{[illegible]} & \text{[illegible]} & \text{[illegible]}
\end{bmatrix}$$

and

$$
\underline{g}(t) = \frac{q}{m} \begin{bmatrix} 0 \\ \delta E_r^e(r,t) - \omega r \delta B_z^e(r,t) - V_z^0 \delta B_\vartheta^e(r,t) \\ \delta E_\vartheta^e(r,t) + V_z^0 \delta B_r^e(r,t) \\ \delta E_z^e(r,t) + \omega r \delta B_r^e(r,t) \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
$$

An appropiate initial condition for this system of equations is any "small" perturbation from the equilibrium solution $\underline{u}^0$. Let $\underline{u}$ be any initial condition for the system of equations (3.61) such that

$$
\| \underline{u} - \underline{u}^0 \|_{X} < d
$$

where d is a sufficiently small positive constant. The corresponding initial condition for equation (3.71), for any such $\underline{u}$, then, is given by

$$
\underline{w}^0 = \underline{u} - \underline{u}^0
$$

The Electrostatic Approximation Model

An additional assumption simplifies the single degree of freedom model considerably:

A8. perturbed self magnetic field effects are negligible

This assumption is occasionally invoked in the study of plasmas and is referred to as the "electrostatic approximation" (Davidson, 1974: 42). The linear model implied by assumptions (A1) through (A7) and the electrostatic approximation, assumption (A8), is now developed.

Assumption (A8) eliminates the equations for $w_8(t)$ and $w_9(t)$ (see equation (3.71)) immediately since these represent (approximations to) the time derivatives of $\delta B_\theta^s(r,t)$ and $\delta B_z^s(r,t)$, respectively. Furthermore, the equations for $w_6(t)$ and $w_7(t)$ are eliminated in the electrostatic approximation. This is justified as follows. Consider equation (3.28) for the perturbed electric field $\underline{\delta E}^e(\underline{x},t)$:

$$-\varepsilon_{ijk}\frac{\partial}{\partial x_j}\delta E_k^s(\underline{x},t) = \frac{\partial}{\partial t}\delta B_i(\underline{x},t) = 0 \qquad (i = 1,2,3)$$

Since the curl of the perturbed field vanishes, it must be expressible as the gradient of a scalar field (Marion, 1965: 105-108):

$$\delta E_i(\underline{x},t) = -\frac{\partial}{\partial x_i}\phi(\underline{x},t) \qquad (i = 1,2,3)$$

Expression of this result in the cylindrical coordinate system and applying assumptions (A5) and (A6) yields

$$\delta E_\theta(r,t) = \delta E_z(r,t) = 0$$

Consequently, the equations involving $w_\theta(t)$ and $w_z(t)$ can be neglected in the system of equations (3.71).

As a result of the foregoing, the linear single degree of freedom model of the last section simplifies to

$$\frac{\partial}{\partial t}\begin{bmatrix} \delta n(r,t) \\ \delta V_r(r,t) \\ \delta V_\theta(r,t) \\ \delta V_z(r,t) \\ \delta E_r(r,t) \end{bmatrix} = \begin{bmatrix} 0 & -n^0 \tilde{D}_r & 0 & 0 & 0 \\ 0 & 0 & \omega_c - 2\omega & -\dfrac{\omega_p^2 V_z^0}{2c^2} r & \dfrac{q}{m} \\ 0 & 2\omega - \omega_c & 0 & 0 & 0 \\ 0 & \dfrac{\omega_p^2 V_z^0}{2c^2} r & 0 & 0 & 0 \\ 0 & -\omega_c \text{[illegible]} r & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \delta n(r,t) \\ \delta V_r(r,t) \\ \delta V_\theta(r,t) \\ \delta V_z(r,t) \\ \delta E_r(r,t) \end{bmatrix}$$

$$+ \frac{q}{m}\begin{bmatrix} 0 \\ \delta E_r^1(r,t) + \text{[illegible]}(r,t) - V_z \text{[illegible]}(r,t) \\ \text{[illegible]} \\ \text{[illegible]} \\ 0 \end{bmatrix} \qquad \text{[illegible]}$$

[illegible]

Assumptions E1 through E5 and A1 through A8 have been introduced in order to develop a sufficiently simple linear model for a demonstration of semigroup theory techniques. These techniques are not dependent upon the numerous assumptions invoked above. The rigid rotor equilibrium is merely one of many equilibrium solutions; in fact, numerically generated equilibria can be developed for a different set of assumptions than E1 through E5. The fundamental problems to be addressed in any infinite-dimensional system remain the same, however. One must demonstrate that the model, with appropriately chosen function spaces, is well posed. This is equivalent to showing the operator A in the abstract Cauchy problem (equations (2.1), (2.2)) is in one of the $\mathfrak{C}$-classes described in Chapter II. In Chapter IV, the electrostatic approximation model is used to illustrate the theory. Specifically, appropriate spaces are chosen and an analytic solution is obtained. In more complicated models, numerical methods will generally be required and determination of appropriate spaces will undoubtedly be more difficult, but the basic principles remain the same.

Conclusion

Models of the dynamic behavior of a charged particle beam have been developed which vary widely in complexity. The most accurate models consist of the microscopic descriptions and involve six independent space-like variables. Macroscopic descriptions are less complex and involve at most three independent spatial variables, but require additional assumptions. Linearizations of both types of descriptions yield models with the abstract Cauchy problem structure.

A single degree of freedom linear perturbation model has been developed based on a physically reasonable equilibrium solution. This model is novel in that it incorporates the effects of the external fields as controls, and it is expressed as an abstract Cauchy problem. As a result, a new particle beam model is now available to researchers in the control community in a form which is directly useful for further analysis.

The final result of this chapter has been the development of a particularly simple model which has a closed-form solution. It is valid whenever the perturbed self magnetic field effects are negligible, and this is the situation so long as all perturbed velocity components are sufficiently small. A solution of this model is developed in the following chapter.

IV. Analysis of the Electrostatic Approximation Model

Introduction

An analysis of the differential equations in the electrostatic approximation model is now presented. At the outset, conventional methods of classifying systems of partial differential equations are discussed. The system of equations (3.72) does not fall into any of these conventional classifications, at least by most authors' definitions. In fact, no treatment of systems with this particular structure could be found, by this author, in control theory literature. Consequently, fundamental concepts must be applied to the problem at hand.

To this end, a simple example of a system which is similar to that of equation (3.72) is introduced. This trivial example provides insight as to how one might choose an appropriate underlying Banach space for these types of systems.

Both physical considerations and insight from this example are then used in selecting a Banach space for the electrostatic approximation model. The matrix of operators in equation (3.72) is shown to be the generator of a strongly continuous group on this space. The associated semigroup of operators is then constructed, and a closed-form solution of the homogeneous abstract Cauchy problem associated with equation (3.72) is given. The nonhomogeneous solution follows immediately, in light of Theorem 2.12, for a broad class of inputs.

Conventional System Classifications

Consider the following system of partial differential equations:

$$\frac{\partial}{\partial t}\underline{u}(x,t) = A(x,t)\frac{\partial}{\partial x}\underline{u}(x,t) + \underline{b}(\underline{u};x,t) \tag{4.1}$$

The unknown vector-valued function $\underline{u}$ assumes values in R^n, A is an n^{th}-order matrix-valued function of (x,t), and $\underline{b}$ is a (possibly nonlinear) function of $\underline{u}$ as well as (x,t). Systems of equations with this structure are sometimes classified as hyperbolic, parabolic or elliptic.

Equation (4.1) is called <u>hyperbolic</u> at the point (x,t) if (1) all roots of the polynomial $P(\lambda;x,t)$, defined by

$$P(\lambda;x,t) = \det[\lambda I - A(x,t)]$$

are real and (2) if there exists a full set of linearly independent eigenvectors (Courant and Hilbert, 1962: 425; Garabedian, 1964: 96). Some authors prefer to define system (4.1) as hyperbolic only if the polynomial $P(\lambda;x,t)$ has n distinct roots, while others refer to such a system as "strictly hyperbolic" or "hyperbolic in the narrow sense" (Zachmonoglou and Thoe, 1976: 362). With this minor exception, there is good agreement among authors of partial differential equations texts on the definition of hyperbolic systems. If the matrix $A(x,t)$ is symmetric, then it is well known that a full set of linearly independent eigenvectors exists. Consequently, various treatments of equation (4.1) have been undertaken under this simplifying assumption (Russell, 1978: 647; Fattorini, 1983: 146).

There is far less agreement on the definition of a parabolic system, however. Hellwig (1964: 70) defines system (4.1) to be parabolic if the polynomial $P(\lambda;x,t)$ has precisely ν distinct real roots, where $1 \leq \nu \leq n-1$. Few authors allow such a broad definition, however. Various restrictions are usually imposed on equations with the structure of (4.1) in order to preserve some of the properties of scalar parabolic equations (e.g., the heat equation) (Eidel'man, 1969: 3). As a result of various authors' viewpoints, we have systems of equations which are defined as "parabolic in the sense of Petrovskiy," "parabolic in the sense of Shilov," or "parabolic in the sense of Shirota" (Eidel'man, 1969: 444-453). No universally accepted definition of a parabolic system of partial differential equations has yet emerged.

While there is general agreement on the definition of an elliptic system, many texts on partial differential equations omit any discussion of such systems. Both Hellwig (1964: 70) and Zachmanoglou and Thoe (1976: 362) define the system (4.1) to be elliptic at the point (x,t) if the polynomial $P(\lambda;x,t)$ has no real eigenvalues. Elliptic systems do not often arise in initial value problems (Courant and

Hilbert, 1962: [illegible]6).

The system of equations (3.72) is now considered within the context of the above classifications. The form of this equation is as follows:

$$\partial_t \underline{u}(x,t) = A_1 \partial_x \underline{u}(x,t) + A(x)\,\underline{u}(x,t) + \underline{h}(x,t) \qquad (4.2)$$

where

$$A_1 = \begin{bmatrix} 0 & -n & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & [illegible] & [illegible] & 0 & 0 \\ [illegible] & [illegible] & [illegible] & [illegible] & [illegible] \end{bmatrix}$$

and,

$$
\underline{h}(x,t) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -V_z^0 & -\omega x \\ 0 & 1 & 0 & V_z^0 & 0 & 0 \\ 0 & 0 & 1 & \omega x & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \delta E_r^e(x,t) \\ \delta E_\theta^e(x,t) \\ \delta E_z^e(x,t) \\ \delta B_r^e(x,t) \\ \delta B_\theta^e(x,t) \\ \delta B_z^e(x,t) \end{bmatrix}
$$

Consequently, the polynomial $P(\lambda;x,t)$ is independent of (x,t), and is given by

$$
P(\lambda,t) = \det(\lambda I - A_0) = \lambda^5
$$

It is easily verified that there are only four linearly independent eigenvectors associated with the eigenvalue $\lambda = 0$. Hence, this system is neither hyperbolic nor elliptic. Furthermore, it is not parabolic under any of the definitions mentioned above except for Hellwig's. Unfortunately, in contrast to the hyperbolic case, no extensive treatments of systems of this type have been found in control literature, so equation (4.2) will be analyzed from fundamental concepts. To this end, a simple example of such a parabolic system is useful.

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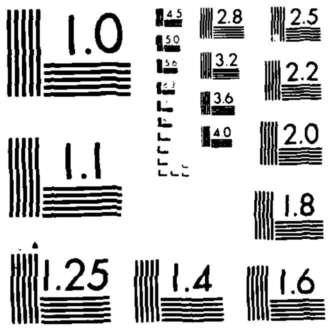

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An Illustrative Example

Consider the two coupled partial differential equations

$$\frac{\partial}{\partial t}u_1(x,t) = -\frac{\partial}{\partial x}u_2(x,t) \tag{4.3}$$

$$\frac{\partial}{\partial t}u_2(x,t) = 0 \tag{4.4}$$

with initial data

$$u_1(x,0) = u_1^0(x) \tag{4.5}$$

$$u_2(x,0) = u_2^0(x) \tag{4.6}$$

for $0 \leq x \leq 1$, $t \geq 0$. Equations (4.3), (4.4) have the structure of equation (4.1) with $\underline{b}(\underline{u};x,t) = \underline{0}$, and

$$A(x,t) = A = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}$$

Note that A has the single eigenvalue $\lambda = 0$, and there is but on linearly independent eigenvector associated with this eigenvalue. The following two propositions are now proven:

Proposition 1: The abstract Cauchy problem of equations (4.3) - (4.6) is not well posed for $X_1 = X_2 = L^2(0,1)$.

Proposition 2: The abstract Cauchy problem of equations (4.3) - (4.6) is well posed for $X_1 = L^2(0,1)$, $X_2 = H^2(0,1)$.

Proof of Proposition 1

Analysis of the operator

$$A = \begin{bmatrix} 0 & -\frac{d}{dx} \\ 0 & 0 \end{bmatrix}$$

reveals that, for $z \neq 0$,

$$(zI - A)^{-1}\underline{g} = \begin{bmatrix} \frac{1}{z} & -\frac{1}{z^2}\frac{d}{dx} \\ 0 & \frac{1}{z} \end{bmatrix} \underline{g}$$

Now $(zI - A)^{-1}$ is not in the set $\mathcal{B}(\mathfrak{X})$ for any value of z since, for the specific choice

$$\underline{g}(x) = \begin{bmatrix} 0 \\ x^{\frac{1}{2}} \end{bmatrix} \varepsilon\ \mathfrak{X}$$

one has that

$$(zI - A)^{-1}\underline{g}(x) = \begin{bmatrix} -\frac{x^{-\frac{1}{2}}}{2z^2} \\ \frac{x^{\frac{1}{2}}}{z} \end{bmatrix}$$

However, $x^{-\frac{1}{2}} \notin X_2 = L^2(0,1)$, and so $(zI - A)^{-1} \notin \mathcal{B}(\mathfrak{X})$. Consequently, the spectrum of A is the entire complex plane and, thus, $A \notin \mathfrak{G}(M,\beta)$ for any pair (M,β) (recall Definition 2.3, page II-10). By Theorem 2.2, then, the abstract Cauchy problem (4.3) - (4.6) is not well posed.

Proof of Proposition 2

In this case, $\mathcal{A} \in \mathcal{B}(\mathfrak{X})$ since

$$\|\mathcal{A}f\|_{\mathfrak{X}}^2 = \left\| \begin{bmatrix} -\frac{df}{dx} \\ 0 \end{bmatrix} \right\|_{\mathfrak{X}}^2 = \left\| \frac{d}{dx} f_2 \right\|_{X_1}^2$$

$$\leq \|f_1\|_{X_1}^2 + \|f_2\|_{X_1}^2 + \left\| \frac{d}{dx} f_2 \right\|_{X_1}^2$$

$$= \|f_1\|_{X_1}^2 + \|f_2\|_{X_2}^2 = \|\underline{f}\|_{\mathfrak{X}}^2$$

By Theorem 2.6, then $\mathcal{A}$ generates a strongly continuous group (which is also a strongly continuous semigroup for $t \geq 0$). Thus, by Theorem 2.2, the abstract Cauchy problem (4.3) - (4.6) is well posed.

By altering the space X_2 from $L^2(0,1)$ to $H^1(0,1)$, then this parabolic system is transformed from an ill-posed to a well-posed problem. Furthermore, for $\mathcal{A} = \mathcal{A}_{\frac{d}{dx}}$, this choice of X_2 yields $\mathcal{A} \in \mathcal{B}(\mathfrak{X})$, and consequently, the associated strongly continuous group $S(t)$ is easily computed:

$$S(t) = \lim_{n\to\infty} \sum_{j=0}^{n} \frac{(t\mathcal{A})^j}{j!} = I + t\mathcal{A} = \begin{bmatrix} 1 & -t\frac{d}{dx} \\ 0 & 1 \end{bmatrix} \quad -\infty < t < \infty$$

The solution, then, of equations (4.3) - (4.6) is

$$z(x,t) = S(t)z^0 \qquad (0 \leq x \leq 1, \ t \geq 0)$$

[illegible]

[illegible]

[illegible]

Solution of the Electrostatic Approximation Model

The electrostatic approximation model developed in Chapter III (see equation (3.72)) is a nonhomogeneous abstract Cauchy problem:

$$\frac{d}{dt}\underline{w}(t) = A\underline{w}(t) + \underline{g}(t) \tag{4.7}$$

$$\underline{w}(0) = \underline{w}^0 \tag{4.8}$$

where

$$A = \begin{bmatrix} 0 & -n^0\tilde{D}_x & 0 & 0 & 0 \\ 0 & 0 & a_{23} & a_{24}x & a_{25} \\ 0 & -a_{23} & 0 & 0 & 0 \\ 0 & -a_{24}x & 0 & 0 & 0 \\ 0 & a_{52} & 0 & 0 & 0 \end{bmatrix}$$

The symbols a_{ij} represent constants in the matrix of operators in equation (3.72), and the operator $\tilde{D}_x$ is defined on page III-31. The mapping $\underline{w}$ takes values from the nonegative real line into a function space $\mathcal{X} = \prod_{i=1}^{5} X_i$ — i.e.,

$$
\underline{w}(t) = \begin{bmatrix} \delta n(x,t) \\ \delta V_r(x,t) \\ \delta V_\theta(x,t) \\ \delta V_z(x,t) \\ \delta E_r(x,t) \end{bmatrix} \in \mathfrak{X} \qquad (t \geq 0)
$$

Analysis of the illustrative example above, equations (4.3) - (4.6), suggests that the choice of $\mathfrak{X}$ has a profound effect on the well-posedness of such systems. Specification of the X_i is now made based upon both the physics of the problem and mathematical insight obtained from the example.

If $n(r,t)$ denotes the number density in a cylindrical beam of radius R with axial and azimuthal symmetry, then the total number of particles in a unit length of the beam at time t, $N(t)$, is given by

$$
N(t) = 2\pi \int_0^R n(r,t)\,r\,dr
$$

The number density can be expressed as an equilibrium value, $n^0(r)$, plus a perturbed number density $\delta n(r,t)$, and thus,

$$
N(t) = 2\pi \int_0^R n^0(r)\,r\,dr + 2\pi \int_0^R \delta n(r,t)\,r\,dr
$$

One natural choice for the norm of the perturbation $\delta n(r,t)$, then, is the $L^1(0,R)$ norm of $r\delta n(r,t)$:

$$\left\| r\delta n(r,t) \right\|_{L^1(0,R)} = \int_0^R \left| \delta n(r,t) r \right| dr$$

Define $M^1(0,R)$ to be the space consisting of all functions g for which

$$\left\| g(x) \right\|_{M^1(0,R)} = \left\| x g(x) \right\|_{L^1(0,R)}$$

In Appendix C it is proven that $M^1(0,R)$ is a Banach space. The space X_1 is now defined to be $M^1(0,R)$.

The Sobolev space $H^1(0,R)$ is selected for the spaces X_2 through X_5. Unlike the choice for X_1, this selection is motivated more by the results of the example in the last section than by physical considerations.

Since the spaces X_1 through X_5 chosen above are all Banach spaces, the Cartesian product $\mathfrak{X} = \prod_{i=1}^{5} X_i$ becomes a Banach space. For convenience, define the norm of the space $\mathfrak{X}$ by

$$\left\| g \right\|_{\mathfrak{X}} = \max_{i=1,\ldots 5} \left\{ \left\| g_i \right\|_{X_i} \right\}$$

With this choice for $\mathfrak{X}$, the linear operator A in the electrostatic approximation model can be shown to be a bounded operator.

Lemma

Let $\mathfrak{X} = \prod_{i=1}^{5} X_i$, with $X_1 = M^1(0,R)$, $X_2 = X_3 = X_4 = X_5 = H^1(0,R)$, and define the operator $A:\mathfrak{X}\to\mathfrak{X}$ by

$$A\underline{f} = \begin{bmatrix} 0 & -n^2\tilde{D}_x & 0 & 0 & 0 \\ 0 & 0 & a_{23} & a_{24}x & a_{25} \\ 0 & -a_{23} & 0 & 0 & 0 \\ 0 & -a_{24}x & 0 & 0 & 0 \\ 0 & a_{52} & 0 & 0 & 0 \end{bmatrix}$$

If $R<\infty$ then $A\in B(\mathfrak{X})$.

Proof

In order to show that $A\in B(\mathfrak{X})$, it is sufficient to show that

$$\|A\underline{f}\| \leq K\|\underline{f}\|$$

for some $K > 0$ (see page II-3). Denoting $A\underline{f}$ by $\underline{g}$, inequalities for $\|\underline{g}\|_{X_i}$ are derived as follows:

$\underline{\|g_1\|_{X_1}}$

$$\|g_1\|_{X_1} = \int_0^R \left| x[-n^{\epsilon}\tilde{D}_x f_2(x)]' \right| dx = n^{\epsilon}\int_0^R \left| f_2(x) + x f_2'(x) \right| dx$$

$$\le n^{\epsilon}\int_0^R |f_2(x)| + |x f_2'(x)| \, dx$$

By the Schwarz inequality,

$$\int_0^R |f_2(x)| \, dx \le \left[\int_0^R dx\right]^{1/2} \left[\int_0^R f_2(x)^2 \, dx\right]^{1/2} = \sqrt{R}\, \|f_2\|_{L^2(0,R)}$$

Similarly,

$$\int_0^R |x f_2'(x)| \, dx \le \left[\int_0^R x^2 \, dx\right]^{1/2} \left[\int_0^R f_2'(x)^2 \, dx\right]^{1/2}$$

$$= \frac{R^{3/2}}{\sqrt{3}} [illegible] (0,R)$$

[illegible]

Consequently,

$$\|g_1\|_{X_1} \leq C_{12}\|f_2\|_{X_2}$$

$\|g_2\|_{X_2}$

$$\|g_2\|_{X_2} = \|a_{23}f_3 + a_{24}xf_4 + a_{25}f_5\|_{X_2}$$

By the triangle inequality, then, and since $X_2 = X_3 = X_4 = X_5$,

$$\|g_2\|_{X_2} \leq |a_{23}|\,\|f_3\|_{X_2} + |a_{24}|\,\|xf_4\|_{X_2} + |a_{25}|\,\|f_5\|_{X_5}$$

The middle term on the right hand side obeys the following inequalities:

$$\|xf_4\|^2_{X_4} = \int_0^R x^2 f_4(x)^2 + [D_x(xf_4(x))]^2 dx$$

$$\leq \int_0^R x^2 f_4(x)^2 + 2[f_4(x)^2 + (xf_4'(x))^2]dx$$

$$\leq \int_0^R 2(x^2+2)[f_4(x)^2 + f_4'(x)^2]dx$$

$$\leq 2(R^2+2)\int_0^R f_4(x)^2 + f_4'(x)^2 dx$$

$$= 2(R^2+2)\|f_4\|^2_{X_4}$$

and so

$$\|xf_4\|_{X_4} \leq [2(R^2+2)]^{\frac{1}{2}}\|f_4\|_{X_4}$$

As a result,

$$\|g_2\|_{X_2} \leqq |a_{23}| \|f_3\|_{X_3} + |a_{24}| \sqrt{2(R^2+2)} \|f_4\|_{X_4} + |a_{25}| \|f_5\|_{X_5}$$

$$= C_{23} \|f_3\|_{X_3} + C_{24} \|f_4\|_{X_4} + C_{25} \|f_5\|_{X_5}$$

$\underline{\|g_3\|_{X_3}}$

$$\|g_3\|_{X_3} = \|-a_{23} f_2\|_{X_3} = |a_{23}| \|f_2\|_{X_2} = C_{32} \|f_2\|_{X_2}$$

$\underline{\|g_4\|_{X_4}}$

$$\|g_4\|_{X_4} = \|-a_{24} x f_2\|_{X_4} \leqq |a_{24}| \sqrt{2(R^2+2)} \|f_2\|_{X_2} = C_{42} \|f_2\|_{X_2}$$

(The argument is identical to that for the $\|xf_4\|_{X_2}$ term above.)

$\underline{\|g_5\|_{X_5}}$

$$\|g_5\|_{X_5} = \|a_{52} f_2\|_{X_5} = |a_{52}| \|f_2\|_{X_2} = C_{52} \|f_2\|_{X_2}$$

From these inequalities, it can be seen that the norm of $A\underline{f}$ satisfies the following:

$$\|A\underline{f}\|_{\mathfrak{X}} = \max_{i=1,\ldots 5}\left\{\|g_i\|_{X_i}\right\}$$

$$\leqq \max\Big\{C_{12}\|f_2\|_{X_2},\ C_{23}\|f_3\|_{X_3} + C_{24}\|f_4\|_{X_4} + C_{25}\|f_5\|_{X_5},$$

$$C_{32}\|f_2\|_{X_2},\ C_{42}\|f_2\|_{X_2},\ C_{52}\|f_2\|_{X_2}\Big\}$$

$$= \max_{i=1,\ldots 5}\left\{\sum_{j=1}^{5} C_{ij}\|f_j\|_{X_j}\right\} \leqq K\|\underline{f}\|_{\mathfrak{X}}$$

The constant K is given by

$$K = \max_{i=1,\ldots 5}\left\{\sum_{j=1}^{5} C_{ij}\right\}$$

and the C_{ij} are either defined as above or are zero if not previously defined. Since A is linear and $\|A\underline{f}\|_{\mathfrak{X}} \leqq K\|\underline{f}\|_{\mathfrak{X}}$, $A \in B(\mathfrak{X})$. ∎

Proof of the following theorem is immediate in light of this lemma and Theorems 2.2 and 2.6:

Theorem 4.1

If the linear operator A and the Banach space $\mathfrak{X}$ are defined as above, then the homogeneous abstract Cauchy problem associated with equation (4.7)

$$\frac{d}{dt}\underline{w}(t) = A\underline{w}(t) \qquad (t \geqq 0)$$

$$\underline{w}(0) = \underline{w}^0$$

is well posed.

Also from Theorem 2.6, the solution of the homogeneous problem is simply

$$\underline{w}(t) = S(t)\underline{w}^0$$

where $S(t)$ is the strongly continuous group generated by A:

$$S(t) = \lim_{n\to\infty} \sum_{j=0}^{n} \frac{t^j A^j}{j!}$$

A closed-form expression for $S(t)$ is now developed.

Computation of $S(t)$ involves the development of general terms in the infinite series indicated above. The work is simplified by the use of a partitioned matrix expression for A:

$$A = \begin{bmatrix} 0 & -n^2\tilde{D}_x & 0 & 0 & 0 \\ 0 & 0 & a_{23} & a_{24}x & a_{25} \\ 0 & -a_{23} & 0 & 0 & 0 \\ 0 & -a_{24}x & 0 & 0 & 0 \\ 0 & a_{52} & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} C & F & 0 \\ 0 & 0 & G \\ 0 & H & 0 \end{bmatrix}$$

Writing out the first few terms of A one obtains noisy environment.

Parameter estimation should also be pursued since some quantities in the model are not likely to be known with great certainty (e.g, V_z^2, [illegible], and n^0). In fact, the relatively new method known as <u>multiple model adaptive control</u> (Maybeck, 1982: 253) could prove useful as well.

<u>Full Linear Macroscopic Model</u>. An examination of the linear macroscopic model of equations (3.66) and (3.67) reveals that, like the electrostatic approximation model, this system of nine partial differential equations is classified as a parabolic system under Hellwig's definition. Writing equation (3.66) in the form of equation (4.1), one has

$$A(x,t) = A = \begin{bmatrix} 0 & -n^0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -c^2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & c^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \end{bmatrix}$$

and, thus,

$$|\lambda I - A| = \lambda^5(\lambda^2 - c^2)^2$$

It is easily verified that only four linearly independent eigenvectors number density, velocity field, and radial electric field, under the assumptions required for the electrostatic approximation model. It is not yet clear whether an arbitrary state can be attained through the action of allowable controls. In the language of control theory, one would like to establish whether the electrostatic approximation model is controllable, approximately controllable (Russell, 1978: 643), or neither.

A second control problem is that of synthesizing a regulator to maintain the equilibrium solution when the model is subjected to unknown (or unmodelled) inputs. Generating a stable configuration for a plasma in the laboratory is frequently difficult. A regulator based on the electrostatic approximation model might improve the situation considerably.

Eventually a controller which would enable changing the state of the beam to a new equilibrium might be sought. Adaptive control would be necessary if the new equilibrium were to be far from the original equilibrium.

Observability Problem. A means of detecting the state of the system is required in order to design a controller. Analytical studies of various sensor configurations can now be performed with the aid of the electrostatic approximation model and its solution.

Deterministic studies are recommended first, to determine general sensor characteristics required in order for the system to be observable or distinguishable (Russell, 1978: 645). Once some general requirements of sensors are determined, stochastic analyses should be performed to determine how well one can estimate the beam state in a (see (Courant and Hilbert, 1962:172,173)). These models do not fit into most classification schemes for systems of first-order partial differential equations: however, they appear to be physically significant. In fact, the electrostatic approximation model has a nonstandard structure, yet a unique solution to this system does exist as is shown in Chapter IV. Perhaps this result indicates a need for a better classification system than presently exists. (Semigroup theory, as applied to the abstract Cauchy problem, may provide insight in this direction.)

Contributions to the field of plasma physics consist of (1) a solution of the electrostatic approximation model (Chapter IV), and, (2) an introduction to (and a demonstration of) the application of semigroup theory to collisionless plasma dynamics problems. The solution of the electrostatic approximation model is a closed-form solution and has not appeared, evidently, previously in the literature. It describes the electromechanical oscillations of a very simple beam dynamics model. Under certain approximating conditions, the beam is shown to oscillate at the plasma frequency as one might expect. The full potential of the techniques employed herein has only begun to be realized in this area of plasma physics.

Suggested Areas for Further Research

Control Problem. Equation (4.10) provides an explicit means of predicting the effects of external electric and magnetic fields on the

V. Summary and Suggested Areas for Further Research

Summary of Research Results

Significant contributions have been made in this research effort to three distinct fields: (1) control theory, (2) applied mathematics, and (3) plasma physics. These contributions are now briefly discussed.

The single most significant accomplishment in this research is the laying of a foundation for application of modern control theory techniques to the beam dynamics problem. This foundation consists of three separate blocks. First, a concise description of relevant semigroup theory results is given. Secondly, a full spectrum of beam dynamics models is developed. Finally, a specific model has been exploited which fully illustrates semigroup theory techniques. The closed-form solution of this model, with external controls included, is in itself significant, but more importantly, the solution process used serves as a pattern for future control theory research efforts.

Two aspects of this research are of interest to applied mathematicians. Development of semigroup theory into a useful tool requires more documented accounts of actual applications: this report represents one additional such account. Another significant result of interest to applied mathematicians is the form of some of the systems of PDE in Chapter III. The structure of some of the models therein is neither totally hyperbolic nor hyperbolic in the more general sense number density, velocity field, and radial electric field evolve in time from an arbitrary initial condition, but it predicts their evolution in the presence of external fields as well. In a broader context, by using solution techniques that involve elements of the semigroup theory of operators, this powerful and elegant theory is now more accessible to both plasma physicists and control theory researchers.

definition, but no extensive techniques for solving such systems exist in the literature. By an application of semigroup theory, and by a careful selection of the underlying Banach space, however, this model was transformed into a well-posed abstract Cauchy problem and a closed-form solution was derived.

One significant conclusion that can be drawn from these developments is that systems of linear partial differential equations can be successfully analyzed by semigroup theory techniques <u>regardless of their conventional classification</u>. Various researchers in this field have recognized this fact. For example, Pazy (1983; 105, 110) classifies equations of the form

$$\frac{d}{dt}\underline{w}(t) = A\underline{w}(t) \qquad (t \geq 0)$$

as either hyperbolic or parabolic depending upon whether A generates a strongly continuous semigroup or an analytic semigroup (defined in (Pazy, 1983: 60)), respectively. Also, Fattorini (1983; 173) classifies equations with this structure as abstract parabolic if every generalized solution of the system is continuously differentiable, and he relates this to the analytic nature of the semigroup. The electrostatic approximation model provides a concrete example of the need for a classification scheme which is based on the properties of the operator A in relation to the underlying Banach space $\mathfrak{X}$.

Several contributions to both plasma physics and control theory have been made in this chapter. In the narrower sense, development of a closed-form solution of the electrostatic approximation model is significant in itself. This solution not only describes how the brackets is essentially unity and the variation of Ω with x can be ignored. Supposing this to be the situation for now,

$$\Omega \simeq [\omega_c^2 - \omega_p^2]^{1/2}$$

In Brillouin flow conditions (see the subsection "Nonrelativistic Rigid-Rotor Equilibrium" in Chapter III), $\omega_c^2 = 2\omega_p^2$, and

$$\Omega \simeq [2\omega_p^2 - \omega_p^2]^{1/2} = \omega_p$$

This represents a limiting situation since the rigid rotor equilibrium can exist only if $\omega_c^2 \geqq 2\omega_p^2$ —, i.e., $\Omega > \omega_p$.

On the other hand, if for some combination of x , V_z^0 and ω_p, Ω does vary substantially with the spatial location, it can still be interpreted as a frequency, but a different frequency of oscillation would exist at each point within the beam.

Conclusion

Conventional methods of classifying systems of partial differential equations were discussed early in this chapter. The electrostatic approximation model was shown to be parabolic under Hellwig's with initial condition $\underline{w}^0$, provided the corresponding components of $\underline{w}^0$ also satisfy Poisson's equation:

$$D_x w_5^0 = \frac{q}{\varepsilon_0} w_1^0$$

Consequently, the initial condition vector $\underline{w}^0$ is constrained by physical condiderations to lie in the subspace Q defined by

$$Q = \{\underline{w}^0 \varepsilon \mathfrak{X} : \tilde{D}_x w_5^0 = \frac{q}{\varepsilon_0} w_1^0\}$$

<u>The Frequency</u> $\underline{\Omega}$. *In the expression for the semigroup* $S(t)$, equation (4.9), the symbol Ω is seen to appear frequently. The physical significance of Ω is now discussed.

In the derivation of $S(t)$, the symbol Ω was defined as

$$\Omega = \left\{ (\omega_c - 2\omega)^2 + \left(\frac{\omega_p^2 V_z^0}{2c^2}\right)^2 x^2 + \omega_p^2 \right\}^{\frac{1}{2}}$$

Using the definition of ω and some algebra one has

$$\Omega = \left\{\omega_c^2 - \omega_p^2\left[1 - \omega_p^2\left(\frac{V_z^0 x}{2c^2}\right)\right]\right\}^{\frac{1}{2}}$$

For any reasonable combination of x, V_z^0 and ω_p, the factor in

For some applications, the requirement for $\underline{g}(t)$ to be continuously differentiable is too strict. Meaning can be given to the expression on the right hand side of equation (4.10) for a much broader class of inputs. For example, even if $\underline{g}$ is in the set $L^1(0,T)$, this expression is termed a "weak" or "mild" solution (Pazy, 1983: 108; Fattorini, 1983: 89).

Comments on the Solution

Two additional topics concerning the solution, the effect of Poisson's equation on the initial conditions, and the physical significance of the frequency Ω, are now discussed.

Initial Conditions. Although the solution, equation (4.10), is correct for any initial condition vector $\underline{w}^0$ in $\mathfrak{X}$, there is a physical restriction on $\underline{w}^0$. Poisson's equation (see equation (3.50))

$$\tilde{D}_x E_r^S(x,t) = \frac{q}{\varepsilon_0} n(x,t)$$

has been used in the rigid rotor equilibrium derivation of $n^0(x)$ and $E_r^0(x)$, but the perturbed quantities $w_1(t) = \delta n(x,t)$, and $w_5(t) = \delta E_r^S(x,t)$ must satisfy Poisson's equation as well:

$$\tilde{D}_x w_5(t) = \frac{q}{\varepsilon_0} w_1(t)$$

It is easily verified that components $w_1(t)$ and $w_5(t)$ obey Poisson's equation, if $\underline{w}(t)$ is the solution of the homogeneous equation

$$
S(t) = \begin{bmatrix}
1 & -n^c \tilde{D}_x\left[\frac{\sin\Omega t}{\Omega}(\cdot)\right] & n^c \tilde{D}_x\left[\frac{(\cos\Omega t-1)}{\Omega^2}a_{23}(\cdot)\right] \\
0 & \cos\Omega t & \frac{a_{23}\sin\Omega t}{\Omega} \\
0 & -\frac{a_{23}\sin\Omega t}{\Omega} & 1+\left(\frac{a_{23}}{\Omega}\right)^2(\cos\Omega t-1) \\
0 & -\frac{a_{24}x\sin\Omega t}{\Omega} & \frac{a_{24}x a_{23}}{\Omega^2}(\cos\Omega t-1) \\
0 & \frac{a_{52}\sin\Omega t}{\Omega} & -\frac{a_{52}a_{23}}{\Omega^2}(\cos\Omega t-1)
\end{bmatrix} \cdots
$$

$$
\begin{bmatrix}
n^c \tilde{D}_x\left[\frac{(\cos\Omega t-1)}{\Omega^2}a_{24}x(\cdot)\right] & n^c \tilde{D}_x\left[\frac{(\cos\Omega t-1)}{\Omega^2}\text{[illegible]}(\cdot)\right] \\
\text{[illegible]} & \text{[illegible]} \\
\text{[illegible]}(\cos\Omega t-1) & \text{[illegible]} \\
1+\left(\text{[illegible]}\right)\text{[illegible]} & \text{[illegible]} \\
\text{[illegible]} & \text{[illegible]}
\end{bmatrix}
$$

Similar results follow for the remainder of the elements of $S(t)$. The complete expression for the strongly continuous group is shown in the figure on the following page. The notation "$(\cdot)$" in the top row is used to indicate that $\tilde{D}_x$ operates on the product of each initial condition component with the expression within the brackets. For example, the (1,2) element of $S(t)$ would operate on w_2^0 as follows:

$$[S(t)]_{12} w_2^0(x) = -n^0 \tilde{D}_x \left[\frac{\sin\Omega t}{\Omega} w_2^0(x)\right]$$

The solution of the homogeneous problem is now complete.

<u>The Nonhomogeneous Solution</u>. Recall from equation (3.72) that

$$\underline{g}(t) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -V_z^0 & -\omega x \\ 0 & 1 & 0 & V_z^0 & 0 & 0 \\ 0 & 0 & 1 & \omega x & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \delta E_r^e(x,t) \\ \delta E_\theta^e(x,t) \\ \delta E_z^e(x,t) \\ \delta B_r^e(x,t) \\ \delta B_\theta^e(x,t) \\ \delta B_z^e(x,t) \end{bmatrix}$$

By Theorem 2.12, if the external fields are such that $\underline{g}(t)$ is continuously differentiable for all $t \varepsilon [0,T]$, then the nonhomogeneous equation (4.7) has the solution

$$\underline{w}(t) = S(t)\underline{w}^0 + \int_0^R S(t-s)\underline{g}(s)\,ds \qquad (0 \leqq t \leqq T) \qquad (4.10)$$

As another example, consider the (1,3), (1,4), and (1,5) elements of $S(t)$:

$$\left[[S(t)]_{13}\ [S(t)]_{14}\ [S(t)]_{15} \right] = \sum_{k=1}^{\infty} \frac{t^{2k}}{(2k)!} FG(HG)^{k-1}$$

$$= FG \sum_{k=1}^{\infty} \frac{t^{2k}}{(2k)!} (GH)^{k-1}$$

$$= \frac{FG}{GH} \sum_{k=1}^{\infty} \frac{t^{2k}}{(2k)!} (GH)^{k}$$

$$= -\frac{FG}{[illegible]} \left[\sum_{k=0}^{\infty} \frac{(-1)^k (\ t)^{2k}}{(2k)!} - 1 \right]$$

$$= -\frac{FG}{[illegible]} (\cos [illegible] t - 1)$$

[illegible] for $k=0$,

$G(H\ )^{k-1}$ [illegible]

[illegible] $k+1$, then,

[illegible] $G(GF)^k$

[illegible]

$$S(t)=I+\sum_{k=1}^{\infty}\frac{t^{2k}}{(2k)!}\begin{bmatrix}0 & 0 & FG(HG)^{k-1}\\ 0 & (GH)^k & 0\\ 0 & 0 & (HG)^k\end{bmatrix}+\sum_{k=0}^{\infty}\frac{t^{2k+1}}{(2k+1)!}\begin{bmatrix}0 & F(GH)^k & 0\\ 0 & 0 & G(HG)^k\\ 0 & H(GH)^k & 0\end{bmatrix}$$

The (1,2) element of $S(t)$ can be summed as follows:

$$[S(t)]_{12} = \sum_{k=0}^{\infty} \frac{t^{2k+1}}{(2k+1)!}F(GH)^k$$

$$= \sum_{k=0}^{\infty} \frac{t^{2k+1}}{(2k+1)!}(-n^0\tilde{D}_x)[-a_{23}^2-(a_{24}x)^2 + a_{25}a_{52}]^k$$

An examination of GH reveals that $GH<0$:

$$GH = -a_{23}^2 - (a_{24}x)^2 + a_{25}a_{52}$$

$$= -(\omega_c-2\omega)^2 - \left[\frac{\omega_p^2 V_z^0}{2c^2}\right]x^2 - \omega_p{}^2$$

Letting $\Omega^2 = -GH$, then, one has

$$(GH)^k = (-1)^k \Omega^{2K}$$

With this form for $(GH)^k$,

$$[S(t)]_{12} = -n^0\tilde{D}_x \sum_{k=0}^{\infty} \frac{(i)^{2k}}{\Omega} \frac{t^{2k+1}}{(2k+1)!} \Omega^{2k+1}$$

$$= -n^0\tilde{D}_x\frac{\sin\Omega t}{\Omega}$$

$$S(t)=I+t\begin{bmatrix}0 & F & 0\\ 0 & 0 & G\\ 0 & H & 0\end{bmatrix}+\frac{t^2}{2!}\begin{bmatrix}0 & 0 & HG\\ 0 & GH & 0\\ 0 & 0 & HG\end{bmatrix}$$

$$+\frac{t^3}{3!}\begin{bmatrix}0 & F(GH) & 0\\ 0 & 0 & G(HG)\\ 0 & H(GH) & 0\end{bmatrix}+\frac{t^4}{4!}\begin{bmatrix}0 & 0 & FGHG\\ 0 & (GH)^2 & 0\\ 0 & 0 & (HG)^2\end{bmatrix}$$

$$+\frac{t^5}{5!}\begin{bmatrix}0 & F(GH)^2 & 0\\ 0 & 0 & G(HG)^2\\ 0 & H(GH)^2 & 0\end{bmatrix}+\frac{t^6}{6!}\begin{bmatrix}0 & 0 & FG(HG)^2\\ 0 & (GH)^3 & 0\\ 0 & 0 & (HG)^3\end{bmatrix}+\ldots$$

(Note that since F is an operator, and G, H are matrices, the order of these factors in the expressions is crucial.) A general term of this expansion can be seen to be

$$\frac{t^j}{j!}\begin{bmatrix}0 & F(GH)^{\frac{j-1}{2}} & 0\\ 0 & 0 & G(HG)^{\frac{j-1}{2}}\\ 0 & H(GH)^{\frac{j-1}{2}} & 0\end{bmatrix}\qquad j = 1,3,5,\ldots$$

$$\frac{t^j}{j!}\begin{bmatrix}0 & 0 & FG(HG)^{\frac{j}{2}-1}\\ 0 & (GH)^{\frac{j}{2}} & 0\\ 0 & 0 & (HG)^{\frac{j}{2}}\end{bmatrix}\qquad j = 2,4,6,\ldots$$

After some manipulation, then, $S(t)$ can be expressed as are associated with the eigenvalue $\lambda = 0$. The space $\mathfrak{X}$ must be chosen with this in mind if the model of equations (3.66) and (3.67) is to be a well posed abstract Cauchy problem. A solution of this model would be beneficial since, unlike the electrostatic approximation model, electromagnetic effects are included. By comparing the solutions of the two models, then, an assessment of the shortcomings of the electrostatic approximation model can be made.

Two Degree of Freedom Model. Although much can be learned from a single degree of freedom model, many of the current particle beam control elements require at least a two degree of freedom model. Analysis of the dynamic behavior of a beam inside a quadrupole with variable magnetic field, for example, could suggest totally new means of beam control. It is recommended that an equilibrium solution be sought for a beam with assumption (A6), the axial symmetry assumption, removed.

System Classification. As was mentioned in Chapter IV, conventional schemes for classifying systems of partial differential equations are not well suited to control applications. It is recommended that further investigation be conducted to establish classifications of such systems based upon both the operator A in the abstract Cauchy problem, equation (2.1), and the underlying Banach space $\mathfrak{X}$.

Appendix A. Mathematical Symbols

Symbols that are frequently used in Chapter II are summarized below. Page numbers are given, following each definition, to indicate where the symbol was first introduced in the text.

$f: A \to B$	a function f which maps each element of set A into an element of set B (II-1)
$\mathcal{D}(f)$, $\mathcal{R}(f)$	domain, range of the function f (II-1)
$\mathcal{R}$, $\mathcal{C}$	real, complex field of scalars (II-2)
$Re(z)$	real part of complex number z (II-10)
R^n, C^n	real, complex, n-fold Cartesian product of $\mathcal{R}$, $\mathcal{C}$ (II-2)
I, Ω	intervals in $\mathcal{R}$, $\mathcal{R}^n$ (II-2)
$\prod_{i=1}^{n} X_i$	Cartesian product of spaces X_i, $i = 1,2,\ldots n$ (II-2)
$L^p(\Omega)$, $H^q(\Omega)$	Lebesgue, Sobolev space of order p, q over interval Ω (II-2)
$\|f\|_X$	norm of f on space X (II-2)
(u, v)	inner product of u, v (II-24)
$B(X,Y)$, $B(X)$	set of all bounded linear operators from X into Y, from X into X (II-3)

$\mathfrak{C}(X,Y)$, $\mathfrak{C}(X)$	set of all closed linear operators from subset of X into Y, from subset of X into X (II-5,6)
$C_0^\infty(\Omega)$	set of all functions which are continuous, have continuous derivatives of all orders, and which have support bounded and contained in Ω (II-6)
$C[a,b]$	set of all functions which are continuous in the sup norm on [a,b] (II-27)
$L^1_{loc}(\Omega)$	set of functions in $L^1(K)$ for every bounded, Lebesgue measurable set K with closure contained in Ω (II-6)
$\frac{d}{dx}$, D_x, $\frac{\partial}{\partial x_k}$, D_{x_k}	ordinary, partial differentiation symbols (generalized derivatives implied unless otherwise stated) (II-6,7)
D^k	$\frac{\partial^{\lvert k \rvert}}{\partial x_1^{k_1} \dots x_n^{k_n}}$, where $\lvert k \rvert = \sum_{i=1}^{n} k_i$, $k=(k_1,\dots k_n)$, with k_i nonnegative integers for $i=1,\dots n$ (II-7)
δF_x, $\hat{\delta} F_x$	Gateaux, Frechet derivative of operator F at x (II-8)
$\rho(A)$	resolvent set of linear operator A (II-9)
$\sigma(A)$	spectrum of linear operator A (II-9)
$R(z,A)$	$(zI-A)^{-1}$ where $z\varepsilon\rho(A)$ (II-10)
$\mathfrak{G}(M,\beta)$, $\mathfrak{G}'(M,\beta)$	see Definition 2.3 (II-10)
"support of f"	the set of points: $\{x: f(x) \neq 0\}$

Appendix B. Physics Symbols

The following symbols are introduced in Chapter III and are summarized here for convenience. Page numbers in parentheses following the definitions indicate where the symbol was first used or defined.

m	particle rest mass (III-4)
q	particle charge (III-4)
c	vacuum speed of light (III-4)
ε_0	absolute dielectric constant (III-8)
μ_0	absolute magnetic permeability of free space (III-8)
ω_c	cyclotron frequency (III-27)
ω_p	plasma frequency (III-27)
ε_{ijk}	$\begin{cases} 1 & (i,j,k) = (1,2,3),\ (2,3,1),\ \text{or}\ (3,1,2) \\ -1 & (i,j,k) = (1,3,2),\ (2,1,3),\ \text{or}\ (3,1,2) \\ 0 & \text{otherwise} \end{cases}$
$\lvert \underline{u} \rvert$	magnitude of vector (III-4)
$\underline{v}$	microscopic velocity vector (III-5)

$\underline{p}$	microscopic momentum vector (III-5)
$n(\underline{x},t)$	number density (III-5)
$\underline{V}(\underline{x},t)$	macroscopic velocity vector (III-5)
$\underline{P}(\underline{x},t)$	macroscopic momentum vector (III-5)
f	distribution function (III-5)
$f^*_{\underline{p}\|\underline{x}}(\underline{p};\underline{x},t)$	$\begin{cases} f(\underline{x},\underline{p},t)/n(\underline{x},t) & n(\underline{x},t) > 0 \\ 0 & n(\underline{x},t) = 0 \end{cases}$ (Page III-7)
$\underline{J}(\underline{x},t)$	current density vector (III-6)
$\underline{E}(\underline{x},t)$	electric field vector (III-4)
$\underline{B}(\underline{x},t)$	magnetic field vector (III-4)
$\mathbb{P}$	pressure tensor (III-6)
T^r_x	transformation matrix (III-19)

Appendix C. Completeness of $M^1(0,R)$

In Chapter IV the space $M^1(0,R)$ is defined as follows:

$$M^1(0,R) = \{f : \|xf\|_{L^1(0,R)} < \infty\}$$

$$\|f\|_{M^1(0,R)} = \|xf\|_{L^1(0,R)}$$

It is asserted in that chapter that $M^1(0,R)$ is a Banach space and this is now proven.

Theorem C.1

The space $M^1(0,R)$ is a Banach space.

Proof

It is obvious that $M^1(0,R)$ is a normed linear space. To show completeness, let $\{g_i\}_1^\infty \subset M^1(0,R)$ be a Cauchy sequence. Defining functions f_i by

$$f_i(x) = xg_i(x) \qquad (0<x<R)$$

it is clear that $\{f_i\}_1^\infty \subset L^1(0,R)$. But this sequence is Cauchy in $L^1(0,R)$ since

$$\|f_m - f_n\|_{L^1(0,R)} = \|x(g_m - g_n)\|_{L^1(0,R)}$$

$$= \|g_m - g_n\|_{M^1(0,R)}$$

and, as $\{g_i\}_1^\infty$ is Cauchy, there exists an N for every $\varepsilon > 0$ such that

$$\|g_m - g_n\|_{M^1(0,R)} < \varepsilon \qquad \forall m,n > N$$

$$\Rightarrow \|f_m - f_n\|_{L^1(0,R)} < \varepsilon \qquad \forall m,n > N$$

Since $L^1(0,R)$ is complete, the sequence $\{f_i\}_1^\infty$ converges to an element f in $L^1(0,R)$. Let a function g be defined by

$$g(x) = \frac{1}{x} f(x) \qquad (0<x<R)$$

Now $g \varepsilon M^1(0,R)$ since

$$\|g\|_{M^1(0,R)} = \|x(\tfrac{1}{x} f)\|_{L^1(0,R)} = \|f\|_{L^1(0,R)}$$

The sequence $\{g_i\}_1^\infty$, then, converges to g since

$$\|g_i - g\|_{M^1(0,R)} = \|x(g_i - g)\|_{L^1(0,R)} = \|f_i - f\|_{L^1(0,R)}$$

and $f_i \to f$ in the $L^1(0,R)$ norm. Thus, every Cauchy sequence in $M^1(0,R)$ converges to an element in $M^1(0,R)$.

Appendix D. A Three Degree-of-Freedom Linear Model

As stated in the footnote on page III-17, the linearization process is in no way limited to a single degree-of-freedom model. By applying the same techniques used in Chapter III to the system of equations on pages III-20 and III-21, with the rigid rotor equilibrium solution (equation 3.60), one can derive a linear model with the following form:

$$\frac{d}{dt}\underline{w}(t) = A\underline{w}(t) + \underline{g}(t)$$

The symbol $\underline{w}(t)$ is given by

$$\underline{w}(t) = \begin{bmatrix} \delta n(\underline{r},t) \\ \delta V_r(\underline{r},t) \\ \delta V_\theta(\underline{r},t) \\ \delta V_z(\underline{r},t) \\ \delta E_r(\underline{r},t) \\ \delta E_\theta(\underline{r},t) \\ \delta E_z(\underline{r},t) \\ \delta B_r(\underline{r},t) \\ \delta B_\theta(\underline{r},t) \\ \delta B_z(\underline{r},t) \end{bmatrix} \quad ; \ \underline{r} = (r,\theta,z)$$

The operator A is expressed in detail on the following page, and

$$
A = \begin{bmatrix}
(\omega - V_z^0) D_\theta & -n^0 \tilde{D}_r & -\frac{n^0}{r} D_\theta & -n^0 D_z & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \omega D_\theta - V_z^0 D_z & \omega_c - 2\omega & -\frac{\omega_p^2 V_z^0 r}{2c^2} & \frac{q}{m} & 0 & 0 & 0 & -\frac{qV_z^0}{m} & -\frac{q\omega r}{m} \\
0 & 2\omega - \omega_c & -\omega D_\theta & -V_z^0 D_z & 0 & \frac{q}{m} & 0 & \frac{qV_z^0}{m} & 0 & 0 \\
0 & \frac{\omega_p^2 V_z^0 r}{2c^2} & 0 & \omega D_\theta - V_z^0 D_z & 0 & 0 & \frac{q}{m} & \frac{q\omega r}{m} & 0 & 0 \\
0 & -\mu_0 q c^2 n^0 & 0 & 0 & 0 & 0 & 0 & 0 & -c^2 D_z & \frac{c^2}{r} D_\theta \\
\mu_0 q c^2 \omega r & 0 & -\mu_0 q c^2 n^0 & 0 & 0 & 0 & 0 & c^2 D_z^0 & 0 & -c^2 D_r \\
-\mu_0 q c^2 V_z^0 & 0 & 0 & -\mu_0 q c^2 n^0 & 0 & 0 & 0 & -\frac{c^2}{r} D_\theta & \frac{c^2}{r} \tilde{D}_r & 0 \\
0 & 0 & 0 & 0 & 0 & D_z & -\frac{1}{r} D_\theta & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -D_z & 0 & D_r & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -\frac{1}{r} D_\theta & -\tilde{D}_r & 0 & 0 & 0 & 0
\end{bmatrix}
$$

$$\underline{g}(t) = \frac{q}{m} G(r)\underline{u}(t)$$

$$= \frac{q}{m} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -V_z^0 & -\omega r \\ 0 & 1 & 0 & V_z^0 & 0 & 0 \\ 0 & 0 & 1 & \omega r & 0 & 0 \\ & & & \cdots & & \\ & & & 0 & & \end{bmatrix} \begin{bmatrix} \delta E_r^e(\underline{r},t) \\ \delta E_\theta^e(\underline{r},t) \\ \delta E_z^e(\underline{r},t) \\ \delta B_r^e(\underline{r},t) \\ \delta B_\theta^e(\underline{r},t) \\ \delta B_z^e(\underline{r},t) \end{bmatrix}$$

Note that the form of the differential equation above is identical to that of equation (3.71), the single degree-of-freedom case. The vector $\underline{w}(t)$ is an element of a function space $\mathfrak{X} = \prod_{i=1}^{10} X_i$, where each X_i is an appropriately defined space of functions defined on a subset of R^3. The dimension of the space is now ten rather than nine, since the perturbed radial magnetic field is identically zero in the single degree-of-freedom case.

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SECURITY CLASSIFICATION OF THIS PAGE

AD-A159 247

REPORT DOCUMENTATION PAGE

1a. REPORT SECURITY CLASSIFICATION: Unclassified

1b. RESTRICTIVE MARKINGS: None

2a. SECURITY CLASSIFICATION AUTHORITY:

2b. DECLASSIFICATION/DOWNGRADING SCHEDULE:

3. DISTRIBUTION/AVAILABILITY OF REPORT: Approved for public release; distribution unlimited

4. PERFORMING ORGANIZATION REPORT NUMBER(S): AFIT/DS/ENC/85-1

5. MONITORING ORGANIZATION REPORT NUMBER(S):

6a. NAME OF PERFORMING ORGANIZATION: School of Engineering, AF Institute of Technology

6b. OFFICE SYMBOL (If applicable): AFIT/EN

7a. NAME OF MONITORING ORGANIZATION:

6c. ADDRESS (City, State and ZIP Code):

7b. ADDRESS (City, State and ZIP Code):

8a. NAME OF FUNDING/SPONSORING ORGANIZATION: A.F. Weapons Laboratory

8b. OFFICE SYMBOL (If applicable):

9. PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER:

8c. ADDRESS (City, State and ZIP Code): Kirtland AFB, NM

10. SOURCE OF FUNDING NOS.

PROGRAM ELEMENT NO.	PROJECT NO.	TASK NO.	WORK UNIT NO.
N/A			

11. TITLE (Include Security Classification): See block 19 below

12. PERSONAL AUTHOR(S): Max A. Stafford, Major, USAF

13a. TYPE OF REPORT: PhD Dissertation

13b. TIME COVERED: FROM ______ TO ______

14. DATE OF REPORT (Yr., Mo., Day): 1984, December

15. PAGE COUNT:

16. SUPPLEMENTARY NOTATION:

17. COSATI CODES

FIELD	GROUP	SUB. GR.

18. SUBJECT TERMS (Continue on reverse if necessary and identify by block number)

19. ABSTRACT (Continue on reverse if necessary and identify by block number)

Blk 11: "Analysis of the Dynamic Behavior of an Intense Charged Particle Beam Using the Semigroup Approach"

Dissertation Advisor and Committee Chairman: David R. Audley, Major, USAF

Approved for public release: IAW AFR 190-17.
Lynn Wolaver 19 Aug 85
LYNN E. WOLAVER
Dean for Research and Professional Development
Air Force Institute of Technology (ATC)
Wright-Patterson AFB OH 45433

20. DISTRIBUTION/AVAILABILITY OF ABSTRACT

UNCLASSIFIED/UNLIMITED ☐ SAME AS RPT. ☐ DTIC USERS ☐

21. ABSTRACT SECURITY CLASSIFICATION:

22a. NAME OF RESPONSIBLE INDIVIDUAL:

22b. TELEPHONE NUMBER (Include Area Code):

22c. OFFICE SYMBOL:

DD FORM 1473, 83 APR

EDITION OF 1 JAN 73 IS OBSOLETE.

SECURITY CLASSIFICATION OF THIS PAGE

Dynamic models of a charged particle beam subject to external electromagnetic fields are cast into the abstract Cauchy problem form. Various applications of intense charged particle beams, i.e., beams whose self electromagnetic fields are significant, might require, or be enhanced by, the use of dynamic control constructed from suitably processed measurements of the state of the beam. This research provides a mathematical foundation for future engineering development of estimation and control designs for such beams.

Beginning with the Vlasov equation, successively simpler models of intense beams are presented, along with their corresponding assumptions. Expression of a model in abstract Cauchy problem form is useful in determining whether the model is well posed. Solutions of well-posed problems can be expressed in terms of a one-parameter semigroup of linear operators. (The state transition matrix for a system of linear, ordinary, first-order, constant coefficient differential equations is a special case of such a semigroup.) The semigroup point of view allows the application of the rapidly maturing modern control theory of infinite-dimensional systems.

An appropriate underlying Banach space is identified for a simple, but nontrivial, single degree of freedom model (the "electrostatic approximation model"), and the associated one-parameter semigroup of linear operators is characterized.

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